

Ω -Admissible Theory II:**New metrics on determinant of cohomology****And****Their applications to moduli spaces of punctured Riemann surfaces****Lin Weng**

Department of Mathematics, Graduate School of Science, Osaka University,
 Toyonaka, Osaka 560, Japan

Abstract. For singular metrics, Ray and Singer's analytic torsion formalism cannot be applied. Hence we do not have the so-called Quillen metric on determinant of cohomology with respect to a singular metric. In this paper, we introduce a new metric on determinant of cohomology by adapting a totally different approach. More precisely, by strengthening results in the first paper of this series, we develop an admissible theory for compact Riemann surfaces with respect to singular volume forms, with which the arithmetic Deligne-Riemann-Roch isometry can be established for singular metrics. As an application, we prove the Mumford type fundamental relations for metrized determinant line bundles over moduli spaces of punctured Riemann surfaces. Moreover, using an idea of D'Hoker-Phong and Sarnak, we introduce a natural admissible metric associated to a punctured Riemann surface via the Arakelov-Poincaré volume, a new invariant for a punctured Riemann surface. With this admissible metric, we make an intensive yet natural study on two Kähler forms on the moduli space of punctured Riemann surfaces associated to the Weil-Petersson metric and the Takhtajan-Zograf metric (defined by using Eisenstein series. Among others, we, together with Fujiki, show that the Takhtajan-Zograf Kähler form is indeed the first Chern form of a certain metrized line bundle). All this finally leads to a more geometric interpretation of our new determinant metrics in terms of special values of Selberg zeta functions. We end this paper by proposing an arithmetic factorization in terms of Weil-Petersson metrics, cuspidal metrics and Selberg zeta functions, which then serves as the most global picture for viewing Riemann surfaces.

Contents

§1. Introduction

§2. ω -Arakelov metrics and ω -intersection theory

§3. ω -Riemann-Roch metric and its properties

§4. ω -Faltings metric

§5. New metrics on determinant of cohomology for singular metrics and Mumford type isometries

Appendix to §5. Universal Riemann-Roch Isomorphism

§6. Arakelov-Poincaré volume and a geometric interpretation of our new metrics

§7. On Takhtajan-Zograf metric over moduli space of punctured Riemann surfaces

Appendix: Arithmetic Factorization Theorem in terms of Intersection

§A1. Degeneration of Weil-Petersson metrics

§A2. Arithmetic Factorization Theorem: a proposal

§1. Introduction

Over a compact Riemann surface, for any (smooth) Hermitian line bundle, with respect to any (smooth) volume form, we may introduce the Quillen metric ([Qu]) on the corresponding determinant of cohomology. Essentially, this is because there exists only discrete spectrum for the associated Laplacian, so that the Ray-Singer's zeta function formalism ([RS]) can be applied. By using Quillen metrics, we then have the so-called Deligne-Riemann-Roch isometry, or equivalently, the Riemann-Roch and the Noether isometries ([De2]).

On the other hand, we cannot apply the same strategy to compact Riemann surfaces with respect to singular volume forms, or better, to punctured Riemann surfaces, due to the fact that a certain continuous spectrum exists for the corresponding Laplacian. Even though, with respect to hyperbolic metrics on Riemann surfaces of finite volume, along with the same line as compact Riemann surfaces, we now have the works done by Efrat ([Ef]), Jorgenson-Lundelius ([JL1], [JL2]), and Takhtajan-Zograf ([TZ1], [TZ2]) on special values of Selberg zeta functions,

regularized determinants of Laplacians, and Quillen metrics, previously it remains to be a very challenging problem to deduce a general but natural theory from them.

Nevertheless, in this paper, we use a quite independent approach to offer a reasonable metric theory for punctured Riemann surfaces. Roughly speaking, we take the Riemann-Roch and Noether isometries as the motivation and hence as the final goal for developing such a theory, since we believe that a good metric theory for punctured Riemann surfaces should ultimately provide us these two isometries in a natural way. Put this in a more practical term, we go as follows.

As stated above, the up-most main difficulty for doing arithmetic for singular metrics is the unpleasant presence of the continuous spectrum for the associated Laplacian. We solve this by developing a general admissible theory with respect to a possibly singular volume form ω , which strengthens the results in our previous paper [We1] in an essential way: we not only deal with points at finite places, where the metric is finite and smooth, we also develop a system to deal with the cusps, where the metric is singular. (Please see (2.3.1), (2.3.2), (2.4.1), (2.5.1) and (2.5.2) for more details.) Similarly as in [We1], the key points at this stage are the existence of the so-called ω -Arakelov metric and various versions of the Mean Value Lemma, which simply claims that even though we start with totally independent, possibly singular, volume forms, the corresponding admissible theories are essentially the same. (Please see Proposition 2.5.1, Proposition 2.5.3, Proposition 3.3.1 and Corollary 5.1.2 for more details.)

To apply the general admissible theory to singular hyperbolic metrics, we then encounter with the second main difficulty: there exists no geometrically natural admissible metric on the canonical line bundle. Recall that the singular hyperbolic metric is natural only when we view it as a metric on the logarithmic canonical line bundle, which consists of the canonical line bundle and the cuspidal line bundle. By the obvious reason, the naive metric on the cuspidal line bundle resulting only the associated Dirac symbol is useless for our arithmetic and geometric consideration: from such a naive metric on the cuspidal line bundle, we cannot get any admissible metric on the canonical line bundle via the decomposition of the original singular hyperbolic metric on the logarithmic canonical line bundle; while without using admissible metrics on the canonical line bundle, it is impossible to apply the general admissible theory. We overcome this by introducing an invariant called Arakelov-Poincaré volume for a (punctured) Riemann surface, (please see (6.1.8) for more details,) which exposes the deep relation between the Euclidean

aspect (induced from the associated Jacobian of its smooth compactification) and the hyperbolic aspect of such a Riemann surface at their disposal – Multiplying the Arakelov metric with respect to the hyperbolic volume form by this invariant, we get a natural admissible metric on the canonical line bundle, which is simply the standard hyperbolic metric when the Riemann surface is compact. (Please see Corollary 6.4.2 and Remark 6.4.1 for more details.) In fact, for compact Riemann surfaces, by using the Mean Value Lemma and a result of D’Hoker-Phong [D’HP] and Sarnak [Sa] on special values of Selberg zeta functions and regularized determinants of hyperbolic Laplacians, such an invariant is first introduced in [We1] to measure the difference between the standard hyperbolic metric and the Arakelov metric with respect to hyperbolic volume form.

As an application to moduli spaces of punctured Riemann surfaces, we give Mumford type fundamental isometries for determinant line bundles equipped with our metrics. (Please see Theorem 5.3.1, Theorem 5.4.1 and Theorem 6.3.1 for more details). As a direct consequence, we show that the Weil-Petersson Kähler form and the Takhtajan-Zograf Kähler form (defined by using Eisenstein series) on the Teichmüller space and on the moduli space of punctured Riemann surfaces with fixed signature naturally arise from our metric theory. Indeed, our study explores the true essence of the pioneer work given by Takhtajan and Zograf [TZ1,2], which motivates our study. Among others, we, together with Fujiki, show that the Takhtajan-Zograf Kähler form is indeed the first Chern form of a certain metrized line bundle. (Please see Theorem 7.3.1 for more details.) All this then leads to a more geometric interpretation of our determinant metric in terms of spectrum theory. (Please see Theorem 7.3.2 and Theorem 6.4.1 for more details.)

Finally, in an appendix, we propose an arithmetic factorization, which is motivated by the results of Masur [Ma] and Wolpert [Wo2] on Weil-Petersson metrics over moduli space of compact Riemann surfaces. The key point here is that the Weil-Petersson metric and the Takhtajan-Zograf metric are algebraic so they are naturally associated to line bundles on the moduli space together with some smooth metrics, the so-called local potentials. On the other hand, such line bundles have natural extensions to the stably compactification of moduli space of Riemann surfaces in the sense of Deligne and Mumford ([DM] and [Kn]), so we may expect that the associated metrics, or clearly, local potentials, admit continuous extensions to the boundary too. Thus by noticing that the Weil-Petersson metric and Takhtajan-Zograf metric are in the nature of arithmetic intersection again, we then

may further expect that the above factorization of line bundles and local potentials to the boundary give us the corresponding line bundles and local potentials associated to the Weil-Petersson metric and the Takhtajan-Zograf metric on the boundary. We anticipate that such a factorization plays a key role in studying the global geometry of Riemann surfaces in future.

As for the language, we intentionally use Deligne pairing [De2], which is certainly a very natural one for our purpose, despite the fact that such a formalism is not as popular as determinant of cohomology.

As a part of my one semester course at Osaka University in 1997/1998, I explained and refined all the results in this paper with the help of Professor Fujiki. I would like to thank Professor Fujiki, Professor Mabuchi and Professor Miyanishi for their supports. Thanks also due to Professor Ueno, Professor Kobayashi and Professor Ohsawa for inviting me to speak on the results of this paper in a series of lectures at the symposium on Arithmetic Geometry and Painlevé Equations, at Nagoya University respectively, due to Professor To for fruitful discussions at the earlier stage of this research, which were unfortunately stopped by some evil force. Finally, I would like to dedicate this paper to Serge Lang.

§2. ω -Arakelov metrics and ω -intersection theory

(2.1) Throughout this paper, we always assume that M^0 is a (punctured) Riemann surface of genus g . Denote its compactification by M , and let $M \setminus M^0 =: \{P_1, \dots, P_N\}$. We will call $P_i, i = 1, \dots, N$, *cusps* of M^0 , and (g, N) the *signature* of M^0 .

Recall that a Hermitian metric ds^2 on M^0 is said to be *of hyperbolic growth near the cusps*, if for each $P_i, i = 1, \dots, N$, there exists a punctured coordinate disc $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ centered at P_i such that for some constant $C_1 > 0$,

$$(i) \quad ds^2 \leq \frac{C_1 |dz|^2}{|z|^2 (\log |z|)^2} \quad \text{on } \Delta^*, \quad (2.1.1)$$

and there exists a local potential function ϕ_i on Δ^* satisfying $ds^2 = \frac{\partial^2 \phi_i}{\partial z \partial \bar{z}} dz \otimes d\bar{z}$ on Δ^* , and for some constants $C_2, C_3 > 0$,

$$(ii) \quad |\phi_i(z)| \leq C_2 \max\{1, \log(-\log |z|)\}, \quad \text{and} \quad (2.1.2)$$

$$(iii) \quad \left| \frac{\partial \phi_i}{\partial z} \right|, \left| \frac{\partial \phi_i}{\partial \bar{z}} \right| \leq \frac{C_3}{|z| |\log |z||} \quad \text{on } \Delta^*. \quad (2.1.3)$$

In this case, we call ds^2 a *quasi-hyperbolic metric*, which is introduced in [TW1]. (See also [Fu], where a general discussion is given. Indeed, one may view quasi-hyperbolic metrics as the realization of good metrics of Mumford [Mu1] in dimension

one.)

For a quasi-hyperbolic metric ds^2 over a punctured Riemann surface M^0 , it follows easily from (2.1.1) that $\text{Vol}(M^0, ds^2) < \infty$. Denote the normalized volume form of ds^2 by ω so that $\text{Vol}(M, \omega) = 1$. In this paper, ω always denotes the normalized volume form on M associated to a smooth metric (on M) or associated to a quasi-hyperbolic metric on M^0 .

(2.2) In [TW 1, Theorem 1], we show that there exists a unique ω -Green's function $g_\omega(\cdot, \cdot)$, or the Green's function with respect to ω , on $M^0 \times M^0 \setminus \{\text{diagonal}\}$, such that the following conditions are satisfied:

(i) For fixed $P \in M^0$, and $Q \neq P$ near P ,

$$g_\omega(P, Q) = -\log |f(Q)|^2 + \alpha(Q),$$

where f is a local holomorphic defining function for P , and α is some smooth function defined near P ;

(ii) $d_Q d_Q^c g_\omega(P, Q) = \omega(Q) - \delta_P$;

(iii) $\int_M g_\omega(P, Q) \omega(Q) = 0$;

(iv) $g_\omega(P, Q) = g_\omega(Q, P)$ for $P \neq Q$;

(v) $g_\omega(P, Q)$ is smooth on $M^0 \times M^0 \setminus \{\text{diagonal}\}$;

(vi) Near each puncture P_i of M , $i = 1, \dots, N$, there exists a punctured coordinate neighborhood Δ^* centered at P_i such that for fixed $Q \in \Delta^*$, there exists a constant $C > 0$ such that

$$|g_\omega(Q, z)| \leq C \max\{1, \log(-\log |z|)\} \quad \text{on } \Delta^*.$$

Here $d_Q^c := \frac{\sqrt{-1}}{4\pi}(\bar{\partial}_Q - \partial_Q)$ is with respect to the second variable (so that $d_Q d_Q^c = \frac{\sqrt{-1}}{2\pi} \partial_Q \bar{\partial}_Q$), and δ_P is the Dirac delta symbol at P .

The proof comes from the following consideration: for the normalized volume form ω associated to a quasi-hyperbolic metric ds^2 over a punctured Riemann surface M^0 , from definition, it is easy to see that there exists a unique locally integrable function β_ω on M such that

$$dd^c \beta_\omega = \omega - \omega_{\text{can}}, \quad \text{and} \quad \int_M \beta_\omega (\omega + \omega_{\text{can}}) = 0. \quad (2.2.1)$$

Here ω_{can} denotes the canonical volume form on M defined as follows: denote by K_M the canonical line bundle of M . On $H^0(M, K_M)$, there exists a natural pairing

$(\phi, \psi) \mapsto \frac{\sqrt{-1}}{2} \int_M \phi \wedge \bar{\psi}$. Fix any orthonormal basis $\{\phi_i\}$ of $H^0(M, K_M)$ with respect to this pairing, by definition,

$$\omega_{\text{can}} := \frac{\sqrt{-1}}{2q} \sum_{j=1}^q \phi_j \wedge \bar{\phi}_j. \quad (2.2.2)$$

Denote by $g(P, Q)$ the Arakelov-Green's function, i.e., the Green's function with respect to ω_{can} . Then we have

Lemma 2.2.1 ([TW1]) *With the same notation as above, the function $g_\omega(P, Q)$ defined on $M^0 \times M^0 \setminus \{\text{diagonal}\}$ by*

$$g_\omega(P, Q) = g(P, Q) + \beta_\omega(P) + \beta_\omega(Q), \quad (2.2.3)$$

satisfies the above conditions (i)~(vi).

Proof. One may prove this lemma as in [La2, Chapter II, Proposition 1.3]. The full details are given in my Osaka lecture notes [We2]. In fact, we only need to remark that with the growth conditions of β and $d\beta$, the arguments in the proof of [La2, Chapter II, Proposition 1.3] involving Stokes' theorem remain valid by considering small circles of radius r centered at the punctures and then letting $r \rightarrow 0$.

(2.3) Now we are ready to define the ω -Arakelov metrics on $\mathcal{O}_M(P)$ for any point $P \in M$ and on K_M , the canonical line bundle of M .

First of all, for any $P \in M^0$, define a metric $\rho_{\text{Ar};\omega;P}$ on $\mathcal{O}_M(P)$ by setting

$$\log \|1_P\|_{\rho_{\text{Ar};\omega;P}}^2(Q) := -g_\omega(P, Q) + \beta_\omega(P) \quad \text{for } Q \neq P \text{ in } M^0. \quad (2.3.1)$$

Here 1_P denotes the defining section of $\mathcal{O}_M(P)$. (Please note in particular that the constant $\beta_\omega(P)$ is added.) Then

$$\begin{aligned} & d_Q d_Q^c(-\log \|1_P\|_{\rho_{\text{Ar};\omega;P}}^2(Q)) \\ &= d_Q d_Q^c(g_\omega(P, Q) - \beta_\omega(P)) \quad (\text{by (2.3.1)}) \\ &= d_Q d_Q^c g_\omega(P, Q) \\ &= \omega(Q) - \delta_P \quad (\text{by 2.2(ii)}) \\ &= \omega(Q) - \delta_{\text{div}(1_P)}. \end{aligned}$$

Hence $c_1(\mathcal{O}_M(P), \rho_{\text{Ar};\omega;P}) = \omega$. Here c_1 denotes the first Chern form.

Secondly, by Lemma (2.2.1) above, we see that

$$-g_\omega(P, Q) + \beta_\omega(P) = -g(P, Q) - \beta_\omega(Q).$$

Thus, **for any point** $\mathbf{P} \in \mathbf{M}$, we (may) define a Hermitian metric $\rho_{\text{Ar};\omega;P}$ on $\mathcal{O}_M(P)$ by setting

$$\log \|1_P\|_{\rho_{\text{Ar};\omega;P}}^2(Q) := -g(P, Q) - \beta_\omega(Q) \quad \text{for } Q \neq P \text{ in } M^0. \quad (2.3.2)$$

In particular, this works also for cusps P_i , $i = 1, \dots, N$. Easily, we see that

$$c_1(\mathcal{O}_M(P), \rho_{\text{Ar};\omega;P}) = \omega. \quad (2.3.3)$$

We will call $\rho_{\text{Ar};\omega;P}$ *the ω -Arakelov metric*, or *the Arakelov metric with respect to ω* , on $\mathcal{O}_M(P)$.

(2.4) A Hermitian line bundle (L, ρ) on M is called ω -admissible, if $c_1(L, \rho) = d(L) \cdot \omega$. Here $d(L)$ denotes the degree of L . From (2.3.3), we have the following

Lemma 2.4.1. *With the same notation as above, $(\mathcal{O}_M(P), \rho_{\text{Ar};\omega;P})$ is ω -admissible.*

Furthermore, by extending $\rho_{\text{Ar};\omega;P}$ linearly on P by using tensor products, we know that over any line bundle L on M , there exist ω -admissible Hermitian metrics, which are parametrized by \mathbb{R}^+ .

For later use, denote $(\mathcal{O}_M(P), \rho_{\text{Ar};\omega;P})$ by $\underline{\mathcal{O}_M(P)}_\omega$, or simply $\underline{\mathcal{O}_M(P)}$ if no confusion arises. If (L, ρ) is an ω -admissible Hermitian line bundle on M , we denote (L, ρ) by \bar{L}^ω or simply \bar{L} by abuse of notation. Similarly, we use $\bar{L}(P)$ to denote $\bar{L} \otimes \underline{\mathcal{O}_M(P)}$.

Thus, in particular, on the canonical line bundle K_M of M , there exist ω -admissible Hermitian metrics. But such metrics are far from being unique. We next make a certain normalization.

On K_M , define *the ω -Arakelov metric* $\rho_{\text{Ar};\omega}$, or *the Arakelov metric with respect to ω* by setting

$$\|h(z) dz\|_{\rho_{\text{Ar};\omega}}^2(P) := |h(P)|^2 \cdot \lim_{Q \rightarrow P} \frac{|z(P) - z(Q)|^2}{e^{-g_\omega(P, Q)}} \cdot e^{-2q\beta_\omega(P)} \quad \text{for } P \in M^0. \quad (2.4.1)$$

Here $h(z) dz$ denotes a section of K_M . Then we see that

$$\|h(z) dz\|_{\rho_{\text{Ar};\omega}}^2(P) = \|h(z) dz\|_{\text{Ar}}^2(P) \cdot e^{(-2q+2)\beta_\omega(P)}. \quad (2.4.2)$$

Here $\|\cdot\|_{\text{Ar}}^2$ denotes the (canonical) Arakelov metric on K_M . Thus by the fact that $\|\cdot\|_{\text{Ar}}^2$ is ω_{can} -admissible, (see e.g. [La2, Chapter IV, Theorem 5.4],) we have

$$\begin{aligned} & c_1(K_M, \rho_{\text{Ar};\omega}) \\ &= (2q - 2)\omega_{\text{can}} + dd^c(-[(-2q + 2)\beta_\omega]) \quad (\text{by (2.4.2)}) \\ &= (2q - 2)\omega_{\text{can}} + (2q - 2)(\omega - \omega_{\text{can}}) \quad (\text{by (2.2.1)}) \\ &= (2q - 2)\omega. \end{aligned}$$

So we have the following

Proposition 2.4.2. *With the same notation as above, $(K_M, \rho_{\text{Ar};\omega})$ is ω -admissible.*

For later use, denote $(K_M, \rho_{\text{Ar};\omega})$ by \underline{K}_{M_ω} , or simply by \underline{K}_M if no confusion arises. Also we denote $(K_M, \rho_{\text{Ar};\omega} \cdot e^{\frac{c}{2}})$ (resp. $\underline{K}_M \otimes \underline{\mathcal{O}}_M(P)$) by \overline{K}_M^c (resp. $\underline{K}_M(P)$) for any constant c .

We end this subsection by giving a geometric interpretation for the ω -Arakelov metric $\rho_{\text{Ar};\omega}$. We begin with a preparation.

Let \bar{L} be an ω -admissible Hermitian line bundle, then for any point $P \in M$, on the restriction $L|_P$, we introduce a metric by multiplying the restriction metric from \bar{L} to P an additional factor $\exp[d(L) \cdot \frac{1}{2}\beta_\omega(P)]$, and we will use the symbol $\bar{L}\|_P$ to indicate the vector space $L|_P$ together with this modification of the metric, and sometimes call it the ω -restriction of \bar{L} at P . With this, by using (2.4.2), (2.2.1), and the fact that the Arakelov metric induces a natural isometry via the residue map $\text{res} : K_M(P)|_P \rightarrow \mathbb{C}$, we see that the Arakelov metric with respect to ω on K_M is the unique metric such that, at each point $P \in M$, the natural residue map res induces the following ω -adjunction isometry

$$\text{res} : \underline{K}_M(P)\|_P \rightarrow \underline{\mathbb{C}}. \quad (2.4.3)$$

Here $\underline{\mathbb{C}}$ denotes the complex plane \mathbb{C} equipped with the ordinary flat metric.

(2.5) For any two line bundles L, L' on M , denote by $\langle L, L' \rangle$ the Deligne pairing associated to L and L' . In this subsection, we define an ω -Deligne norm $h_{\text{De},\omega}$ on $\langle L, L' \rangle$ for any two ω -admissible Hermitian line bundles \bar{L} and \bar{L}' .

First, let us define the ω -Deligne norm for $\langle \mathcal{O}_M(P), \mathcal{O}_M(Q) \rangle$ with $P \neq Q \in M^0$, for ω -Arakelov metrized line bundles $\underline{\mathcal{O}}_M(P)$ and $\underline{\mathcal{O}}_M(Q)$, by setting

$$\log \|\langle 1_P, 1_Q \rangle\|_{h_{\text{De},\omega}}^2 := -g_\omega(P, Q) + \beta_\omega(P) + \beta_\omega(Q). \quad (2.5.1)$$

Secondly, note that the right hand side of (2.5.1) can be written as $-g(P, Q)$, the Arakelov-Green's function for P and Q . Hence, even though (2.5.1) does not make any sense for cusps, but if we change it to

$$\log \|\langle 1_P, 1_Q \rangle\|_{h_{\text{De},\omega}}^2 := -g(P, Q), \quad (2.5.2)$$

then we have the metrized ω -Deligne pairing $\langle \underline{\mathcal{O}}_M(P), \underline{\mathcal{O}}_M(Q) \rangle$ for **all** $P \neq Q \in M$.

Finally extending $h_{\text{De},\omega}$ by linearity, we get a definition for ω -Deligne norm $h_{\text{De},\omega}(\bar{L}, \bar{L}')$ on $\langle L, L' \rangle$ for any two ω -admissible Hermitian line bundles \bar{L} and \bar{L}' on M . By abuse of notation, we denote $\left(\langle L, L' \rangle, h_{\text{De},\omega}(\bar{L}, \bar{L}')\right)$ simply by $\langle \bar{L}, \bar{L}' \rangle$.

Remark 2.5.1. Even though we study the ω -intersection, the Arakelov-Green's function is used in an essential way. This is indeed not quite surprising. After all, we only define the ω -intersection for **the** Hermitian line bundles $\underline{\mathcal{O}_M(P)}$ and $\underline{\mathcal{O}_M(Q)}$ by using $-g(P, Q)$. Put this in a more formal manner, we have the following:

Proposition 2.5.1. (Mean Value Lemma I.) *For any two normalized volume forms ω_1 and ω_2 on M , there exists a natural isometry*

$$\langle \underline{\mathcal{O}_M(P)}_{\omega_1}, \underline{\mathcal{O}_M(Q)}_{\omega_1} \rangle \simeq \langle \underline{\mathcal{O}_M(P)}_{\omega_2}, \underline{\mathcal{O}_M(Q)}_{\omega_2} \rangle \quad \text{for } P \neq Q \in M. \quad (2.5.3)$$

As a direct consequence of the ω -adjunction isometry (2.4.3), by definition, we have the following:

Proposition 2.5.2. (ω -Adjunction Isometry) *With the same notation as above, we have the isometry*

$$\langle \underline{K_M(P)}, \underline{\mathcal{O}_M(P)} \rangle \simeq \mathbb{C} \quad \text{for any } P \in M. \quad (2.5.4)$$

In a similar style, by using (2.2.1) and (2.4.2), we get

Proposition 2.5.3. (Mean Value Lemma II.) *With the same notation as above, for any two normalized volume forms ω_1 and ω_2 on M , there exists a natural isometry*

$$\langle \underline{K_{M_{\omega_1}}}, \underline{K_{M_{\omega_1}}} \rangle \simeq \langle \underline{K_{M_{\omega_2}}}, \underline{K_{M_{\omega_2}}} \rangle. \quad (2.5.5)$$

Proof. We may assume that ω_2 is simply ω_{can} . Denote ω_1 simply by ω . Then

$$\begin{aligned}
& \langle \underline{K}_{M_\omega}, \underline{K}_{M_\omega} \rangle \\
&= \langle \underline{K}_{M_{\omega_{\text{can}}}} \cdot e^{-(2q-2)\beta_\omega}, \underline{K}_{M_\omega} \rangle \\
&\simeq \langle \underline{K}_{M_{\omega_{\text{can}}}}, \underline{K}_{M_\omega} \rangle \cdot e^{-(2q-2) \int \beta_\omega \cdot c_1(\underline{K}_{M_\omega})} \\
&\simeq \langle \underline{K}_{M_{\omega_{\text{can}}}}, \underline{K}_{M_{\omega_{\text{can}}}} \cdot e^{-(2q-2)\beta_\omega} \rangle \cdot e^{-(2q-2) \int \beta_\omega \cdot c_1(\underline{K}_{M_\omega})} \\
&\simeq \langle \underline{K}_{M_{\omega_{\text{can}}}}, \underline{K}_{M_{\omega_{\text{can}}}} \rangle \cdot e^{-(2q-2) \int \beta_\omega \cdot c_1(\underline{K}_{M_{\omega_{\text{can}}})}} \cdot e^{-(2q-2)^2 \int \beta_\omega \cdot \omega} \\
&\simeq \langle \underline{K}_{M_{\omega_{\text{can}}}}, \underline{K}_{M_{\omega_{\text{can}}}} \rangle \cdot e^{-(2q-2)^2 \int \beta_\omega \cdot \omega_{\text{can}}} \cdot e^{-(2q-2)^2 \int \beta_\omega \cdot \omega} \\
&\simeq \langle \underline{K}_{M_{\omega_{\text{can}}}}, \underline{K}_{M_{\omega_{\text{can}}}} \rangle \cdot e^{-(2q-2)^2 \int \beta_\omega \cdot (\omega_{\text{can}} + \omega)} \\
&\simeq \langle \underline{K}_{M_{\omega_{\text{can}}}}, \underline{K}_{M_{\omega_{\text{can}}}} \rangle \cdot e^{-(2q-2)^2 \cdot 0} \\
&\simeq \langle \underline{K}_{M_{\omega_{\text{can}}}}, \underline{K}_{M_{\omega_{\text{can}}}} \rangle.
\end{aligned}$$

This completes the proof.

Remark 2.5.1. The above Mean Value Lemma says that even though we start with totally independent, possibly singular, volume forms, the corresponding admissible intersections are essentially the same.

As an application to arithmetic surfaces, we see that the self-intersection of Arakelov canonical divisor can be understood in any of these ω -admissible theories. (For the detailed discussion, see e.g. [We1].)

§3. ω -Riemann-Roch metric and its properties

(3.1) With the same notation as in §2, for any line bundle L on M , denote its associated determinant of cohomology, i.e., $\det H^0(M, L) \otimes (\det H^1(M, L))^{\otimes -1}$, by $\lambda(L)$. Then it is well-known that we have the following canonical Riemann-Roch isomorphism;

$$\lambda(L)^{\otimes 2} \otimes \lambda(\mathcal{O}_M)^{\otimes -2} \simeq \langle L, L \otimes K_M^{\otimes -1} \rangle. \quad (3.1.1)$$

(See e.g., [De2], or [Ai].)

For a fixed normalized volume form ω on M associated to a quasi-hyperbolic metric, denote by \underline{K}_M the ω -Arakelov canonical line bundle $(K_M, \rho_{\text{Ar}; \omega})$. With respect to \underline{K}_M , fix a metric $h_0(\underline{K}_M)$ on $\lambda(\mathcal{O}_M)$. Then for any ω -admissible Hermitian line bundle \bar{L} on M , define an ω -determinant metric $h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}(\bar{L})$ on $\lambda(L)$ by the isometry

$$\left(\lambda(L), h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}(\bar{L}) \right)^{\otimes 2} \otimes \left(\lambda(\mathcal{O}_M), h_0(\underline{K}_M) \right)^{\otimes -2} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K}_M^{\otimes -1} \rangle. \quad (3.1.2)$$

We call $h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}(\bar{L})$ on $\lambda(L)$ the ω -Riemann-Roch metric associated to \bar{L} with respect to \underline{K}_M and $h_0(\underline{K}_M)$. Since for a fixed \bar{L} , with respect to \underline{K}_M and $h_0(\underline{K}_M)$, both $\left(\lambda(\mathcal{O}_M), h_0(\underline{K}_M)\right)$ and $\langle \bar{L}, \bar{L} \otimes \underline{K}_M^{\otimes -1} \rangle$ are fixed, $h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}(\bar{L})$ is well-defined. By abuse of notation, we denote $\left(\lambda(L), h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}(\bar{L})\right)$ simply by $\lambda(\bar{L})$.

The ω -Riemann-Roch metric satisfies the following properties, which are very similar to these for Faltings metrics. (See Theorem 4.1.1 below.)

Proposition 3.1.1. *With the same notation as above, we have*

(F1) *An isometry of ω -admissible Hermitian line bundles $\bar{L} \rightarrow \bar{L}'$ induces an isometry from $\lambda(\bar{L})$ to $\lambda(\bar{L}')$;*

(F2) *If the ω -admissible metric on L is changed by a factor $\alpha \in \mathbb{R}^+$, then the metric on $\lambda(L)$ is changed by the factor $\alpha^{\chi(M, L)}$;*

(F3) *For any point P on M , put the ω -Arakelov metric on $\mathcal{O}_M(P)$, and take the tensor metric on $L(-P)$. Then the algebraic isomorphism*

$$\lambda(L) \simeq \lambda(L(-P)) \otimes L|_P$$

induced by the short exact sequence of coherent sheaves

$$0 \rightarrow L(-P) \rightarrow L \rightarrow L|_P \rightarrow 0$$

is an isometry

$$\lambda(\bar{L}) \simeq \lambda(\bar{L} \otimes \underline{\mathcal{O}_M(P)}^{\otimes -1}) \otimes \bar{L}|_P.$$

(F4) (Serre Isometry) $\left(\lambda(K_M), h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}(\underline{K}_M)\right) \simeq \left(\lambda(\mathcal{O}_M), h_0(\underline{K}_M)\right)$.

Proof. (F4) is simply the Serre duality. (F1) and (F2) are direct consequence of the definitions of the ω -Riemann-Roch metric and the ω -intersection. Finally, (F3) is a direct consequence of the definition of the ω -Riemann-Roch metric and the ω -adjunction isometry, which also explains why in our definition of the ω -Riemann-Roch metric and the proposition here we use \underline{K}_M and \underline{P} , i.e., K_M and $\mathcal{O}_M(P)$ together with the ω -Arakelov metrics.

Remark 3.1.1. By (F4), we see that giving a normalization for $h_0(\underline{K}_M)$ on $\lambda(\mathcal{O}_M)$ is equivalent to normalizing $h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}$ on $\lambda(K_M)$.

(3.2) Similarly, with respect to \overline{K}_M , i.e., K_M with an arbitrary ω -admissible metric, we fix a metric $h_0(\overline{K}_M)$ on $\lambda(\mathcal{O}_M)$. Then with respect to \overline{K}_M' , i.e., K_M equipped

with (possibly) another ω -admissible Hermitian metric, and $h_0(\overline{K_M})$, for any ω -admissible Hermitian line bundle \bar{L} , we may define the associated Riemann-Roch metric, denoted by $h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L})$, by the isometry

$$\left(\lambda(L), h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L}) \right)^{\otimes 2} \otimes \left(\lambda(\mathcal{O}_M), h_0(\overline{K_M}) \right)^{\otimes -2} \simeq \langle \bar{L}, \bar{L} \otimes (\overline{K_M}')^{\otimes -1} \rangle. \quad (3.2.1)$$

The dependence of $h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L})$ on \bar{L} and $\overline{K_M}'$ is clear, as it is given by the ω -intersection theory. More precisely, directly from the definition, we have

Proposition 3.2.1. *The dependence of $h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L})$ on \bar{L} and $\overline{K_M}'$ is given by the following equality:*

$$h_{\text{RR}; \overline{K_M}' \otimes \mathcal{O}_M(e^c); h_0(\overline{K_M})}(\bar{L} \otimes \mathcal{O}_M(e^f)) = h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L}) \cdot e^{\chi(L) \cdot f - d(L)c/2}. \quad (3.2.2)$$

Here for a constant c , $\mathcal{O}_M(e^c)$ denotes the trivial line bundle equipped with the metric $\|1\|^2 = e^c$.

On the other hand, the dependence of $h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L})$ on $\overline{K_M}$ is not so easy to determined. Indeed, the most essential part for such a dependence is independent of the above (weak) Riemann-Roch isometry. Nevertheless, from our study on the admissible theory with respect to smooth volume forms in [We1], it is very natural to take the following principle, which has its root from the Polyakov variation formula (see e.g., [Fay2, (3.30)]):

$$h_0(\overline{K_M}^c) := h_0(\underline{K_M}) \cdot e^{\frac{2q-2}{12} \cdot c}. \quad (3.2.3)$$

Here, as before, $\overline{K_M}^c = \underline{K_M} \otimes \mathcal{O}_M(e^c)$.

Remark 3.2.1. The reader may ask why we use $2q-2$ instead of using $2q-2+N$ in (3.2.3). We justify our choice by the following observation: the Hermitian metric on K_M used in (3.2.3) would have the first Chern form $(2q-2)\omega$, which is different from the singular metric introduced by the quasi-hyperbolic metric, for which the total volume is $2\pi(2q-2+N)$. So our normalization is in the same spirit as the one in [JL2, §7].

That is, we have the following

Proposition-Definition 3.2.2. (Polyakov Variation Formula I) *With the same notation as above, we have the following relation*

$$h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M} \otimes \mathcal{O}_M(e^c))}(\bar{L}) = h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L}) \cdot e^{\frac{2q-2}{12} \cdot c}. \quad (3.2.4)$$

We end this section by the following consequence of the definition of the ω -Riemann-Roch metric.

Proposition 3.2.3. (Serre Isometry) *With the same notation as above, we get the isometry:*

$$\left(\lambda(L), h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L}) \right) \simeq \left(\lambda(K_M \otimes L^{\otimes -1}), h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\overline{K_M}' \otimes \bar{L}^{\otimes -1}) \right). \quad (3.2.5)$$

(3.3) In (3.1) and (3.2), for a **fixed** normalized volume form ω on M , we introduce $h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L})$ in such a way that if one of $h_0(\overline{K_M}'')$ is fixed, then all other determinant metrics $h_{\text{RR}; \overline{K_M}'; h_0(\overline{K_M})}(\bar{L})$ are fixed, by using (3.2.2) and (3.2.4), or better Proposition 3.2.1 and Proposition 3.2.2.

Now we explain how the ω -Riemann-Roch metrics depend on ω . Similarly, motivated by our work on admissible theory with respect to smooth volume forms in [We1], we relate different ω -Riemann-Roch metrics by using the following isometry: for any two normalized volume forms ω_1 and ω_2 on M ,

$$\left(\lambda(\mathcal{O}_M), h_0(\underline{K_{M_{\omega_1}}}) \right) \simeq \left(\lambda(\mathcal{O}_M), h_0(\underline{K_{M_{\omega_2}}}) \right). \quad (3.3.1)$$

In other words, *even though $\underline{K_{M_{\omega}}}$, the ω -Arakelov canonical line bundle, depends on ω in an essential way, but the induced metric on the determinant of cohomology does not depend on ω at all.* We may say that this is one of the most important discovery in [We1], where we establish this relation for Quillen metrics. As a direct consequence, we get the following;

Proposition 3.3.1. (Mean Value Lemma III) *With the same notation and normalization as above, for any two normalized volume forms ω_1 and ω_2 on M , we get the following isometries:*

(a) (Polyakov Variation Formula II)

$$\frac{\lambda(\underline{K_{M_{\omega_1}}})_{\omega_1}}{\omega_1} \simeq \frac{\lambda(\underline{K_{M_{\omega_2}}})_{\omega_2}}{\omega_2}. \quad (3.3.2)$$

(b) For all $n_j \in \mathbb{Z}$ and $Q_j \in M$,

$$\frac{\lambda(\mathcal{O}_M(\Sigma_j n_j Q_j)_{\omega_1})_{\omega_1}}{\omega_1} \simeq \frac{\lambda(\mathcal{O}_M(\Sigma_j n_j Q_j)_{\omega_2})_{\omega_2}}{\omega_2}. \quad (3.3.3)$$

Proof. The key point here is that on K_M and $\mathcal{O}_M(Q_j)$, all metrics are carefully chosen to be ω -Arakelov metrics. Thus (a) comes from the Serre isometry; while (b) is deduced from the Riemann-Roch isometry and the Mean Value Lemma I for ω -arithmetic intersection by a tedious calculation.

Thus by the above three kinds of normalizations for the ω -determinant metrics, i.e., (3.2.2), (3.2.4) and (3.3.2), we see that in order to get the ω -Riemann-Roch metric uniquely, we have two different ways to normalize them: one is to uniformly define metrics $h_0(\underline{K}_{M_\omega})$ for all normalized volume forms ω first, which satisfy (3.3.2), i.e., Proposition 3.3.1(a), and then use Proposition 3.2.1 and Proposition 3.2.2 to get all other metrics for any admissible line bundles; while the other is to define for all ω -admissible Hermitian line bundle \bar{L} the Riemann-Roch metrics $h_{\text{RR}; \overline{K_M}^\omega; h_0(\overline{K_M}^\omega)}(\bar{L})$ on $\lambda(L)$ for a certain fixed ω , then to check these metrics satisfy (3.3.2) and (3.2.4) and hence is compactible with our theory. We next give two independent approaches to show how this can be possibly done in a very concrete manner. The first is with respect to any normalized volume form, which then gives an alternative way to find Quillen metric; while the second works only for singular hyperbolic volume forms, which then leads to a more geometric interpretation of our new determinant metric.

§4. ω -Faltings metric

(4.1) This approach begins with the following condition.

(F0) *With respect to the normalized volume ω associated to a quasi-hyperbolic metric $d\mu$ on a compact Riemann surface M , the metric $h_{\text{RR}; \underline{K_M}; h_0(\underline{K_M})}$ on $\lambda(K_M)$ is defined to be the determinant of the Hermitian metric on $H^0(M, K_M)$ induced from the following natural pairing*

$$(\phi, \psi) \mapsto \frac{\sqrt{-1}}{2} \int_M \phi \wedge \bar{\psi}. \quad (4.1.1)$$

Remark 4.1.1. It appears that (F0) is quite strange as no ω is involved (in the natural pairing). But one should not understand in this way, as it is obvious that the above natural pairing on $H^0(M, K_M)$ can also be defined by using any metric $d\mu$ on the Riemann surface M due to the fact that the dimension of the base manifold M is one (so that the dual of the tangent bundle is simply the canonical line bundle).

Now we may improve Proposition 3.1.1 as follows.

Theorem 4.1.1. *With respect to the normalized volume ω on a compact Riemann surface M , for any ω -admissible Hermitian line bundle \bar{L} , there exists a unique metric $h_{\text{RR}; \underline{K}_M; h_0(\underline{K}_M)}(\bar{L})$, denoted also by $h_{F; \omega}(\bar{L})$ and called the ω -Faltings metric, on $\lambda(L)$ such that conditions (F0) \sim (F5) are satisfied. Moreover, we have the following Riemann-Roch isometry:*

$$\left(\lambda(L), h_{F; \omega}(\bar{L}) \right)^{\otimes 2} \otimes \left(\lambda(\mathcal{O}_M), h_{F; \omega}(\underline{\mathcal{O}}_M) \right)^{\otimes -2} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K}_M^{\otimes -1} \rangle. \quad (4.1.2)$$

Proof. The proof of this theorem is the same as the one for Faltings' original theorem [Fa, Theorem 1]. Namely, fixed a large enough positive integer r and a degree $r + q - 1$ divisor E on M . Then, for any point $(Q_1, \dots, Q_r) \in M^r$, by a tedious calculation, we get the isometry

$$\begin{aligned} & \left(\lambda(\mathcal{O}_M(E - Q_1 - \dots - Q_r)), h_{F; \omega}(\underline{\mathcal{O}}_M(E - Q_1 - \dots - Q_r)) \right) \\ & \simeq \left(\lambda(\mathcal{O}_M(E)), h_{F; \omega}(\underline{\mathcal{O}}_M(E)) \right) \otimes \otimes_{i=1}^r \left(\underline{\mathcal{O}}_M(E) \parallel_{Q_i} \right)^{\otimes -1} \\ & \otimes \otimes_{1 \leq i < j \leq r} \left(\underline{\mathcal{O}}_M(Q_i) \parallel_{Q_j} \right), \end{aligned}$$

by using (F3). Noticing that in Faltings' original theorem, when dealing with the norm on the restriction, Faltings uses the most direct restriction for the canonical volume form, while for us, we modify it to \parallel , i.e., we use the ω -restriction. Thus after taking dd^c on M^r ,

$$c_1(\underline{\mathcal{O}}_M(E) \parallel_{Q_i}) = (r + q - 1) \text{pr}_i^*(\omega - dd^c \beta_\omega) = (r + q - 1) \text{pr}_i^* \omega_{\text{can}}.$$

Here $\text{pr}_i : M^r \rightarrow M$ denotes the i -th projection. Similarly,

$$c_1(\underline{\mathcal{O}}_M(Q_i) \parallel_{Q_j}) = dd^c \left(g_\omega(Q_i, Q_j) - \beta_\omega(Q_i) \right) - dd^c \beta_\omega(Q_j) = dd^c g(Q_i, Q_j).$$

Thus, we get

$$\begin{aligned} & c_1 \left(\lambda(\mathcal{O}_M(E - Q_1 - \dots - Q_r)), h_{F; \omega}(\underline{\mathcal{O}}_M(E - Q_1 - \dots - Q_r)) \right) \\ & = - (r + q - 1) \sum_{i=1}^r \text{pr}_i^* \omega_{\text{can}} + \sum_{1 \leq i < j \leq r} \left((\text{pr}_i^* \omega_{\text{can}} + \text{pr}_j^* \omega_{\text{can}}) \right. \\ & \quad \left. - \frac{\sqrt{-1}}{2} \sum_{k=1}^q (\text{pr}_i^* \phi_k \wedge \text{pr}_j^* \bar{\phi}_k + \text{pr}_j^* \phi_k \wedge \text{pr}_i^* \bar{\phi}_k) \right), \end{aligned}$$

which is well-known to be the pull-back of the first Chern form of the Theta bundle together with the standard metric induced by using theta norms. (See e.g., (4.3)

below or [La2, p. 146].) Thus, following the standard discussion in [La2, Chapter VI, §2-§3], we complete the proof of the theorem.

Remark 4.1.1 It is not surprising that the above theorem holds for a general ω as by definition we have the following isometries:

$$\underline{\mathcal{O}_M(Q_i)}_\omega \|_{Q_j} \simeq \underline{\mathcal{O}_M(Q_i)}_{\omega_{\text{can}}} \|_{Q_j} (\simeq \underline{\mathcal{O}_M(Q_i)}_{\omega_{\text{can}}} |_{Q_j})$$

and

$$\underline{\mathcal{O}_M(E)}_\omega \|_{Q_i} \simeq \underline{\mathcal{O}_M(E)}_{\omega_{\text{can}}} \|_{Q_i} (\simeq \underline{\mathcal{O}_M(E)}_{\omega_{\text{can}}} |_{Q_i}).$$

Similarly, we have

$$\underline{K_M}_\omega \|_Q \simeq \underline{K_M}_{\omega_{\text{can}}} \|_Q (\simeq \underline{K_M}_{\omega_{\text{can}}} |_Q)$$

for any point $Q \in M$.

(4.2) In this section, we give further properties for the ω -Faltings metrics.

First of all, by definition, we have the following.

Lemma 4.2.1. *With the same notation as above, there exists a natural isometry*

$$\left(\lambda(K_M), h_{F;\omega}(\underline{K_M})_\omega \right) \simeq \left(\lambda(K_M), h_{F;\omega_{\text{can}}}(\underline{K_M}_{\omega_{\text{can}}}) \right). \quad (4.2.1)$$

On the other hand, by the proof of Theorem 4.1.1, we see that, for general points $(Q_1, \dots, Q_q, Q) \in M^{q+1}$ such that $H^0(M, \mathcal{O}_M(Q_1 + \dots + Q_q - Q)) = H^1(M, \mathcal{O}_M(Q_1 + \dots + Q_q - Q)) = \{0\}$, $\lambda(\mathcal{O}_M(Q_1 + \dots + Q_q - Q))$ is simply \mathbb{C} , and the norm 1 in \mathbb{C} is proportional to $\|\theta(Q_1 + \dots + Q_q - Q)\|$, so that the ratio is independent of (Q_1, \dots, Q_q, Q) . Such a ratio gives an invariant associated to (M, ω) . Following Faltings, we define the ω -Faltings delta function $\delta(M, \omega)$ by

$$\|1\|_{h_{F;\omega}(\underline{\mathcal{O}_M(Q_1+\dots+Q_q-Q)})} = e^{-\delta(M;\omega)/8} \|\theta(Q_1 + \dots + Q_q - Q)\|. \quad (4.2.2)$$

Proposition 4.2.2. *With the same notation as above, we have*

$$\delta(M; \omega) = \delta(M; \omega_{\text{can}}) (= \delta(M)). \quad (4.2.3)$$

That is, ω -Faltings delta function $\delta(M; \omega)$ is the same as the original Faltings delta function $\delta(M)$.

Proof. First of all, by (F3), for any point $Q \in M$, we have the natural isometry

$$\left(\lambda(K_M), h_{F;\omega}(\underline{K_M}) \right) \simeq \left(\lambda(K_M(Q)), h_{F;\omega}(\underline{K_M(Q)}) \right)$$

due to the fact that

$$\underline{K_M(Q)}\|Q \simeq \underline{\mathbb{C}}.$$

Secondly, by a tedious calculation using (F3) again, we get the following isometry

$$\begin{aligned} & \left(\lambda(K_M(Q)), h_{F;\omega}(\underline{K_M(Q)}) \right) \\ & \simeq \left(\lambda(K_M(Q - Q_1 - \cdots - Q_q)), h_{F;\omega}(\underline{K_M(Q - Q_1 - \cdots - Q_q)}) \right) \\ & \quad \otimes \otimes_{i=1}^q \underline{K_M}\|_{Q_i} \otimes \otimes_{i=1}^q \underline{\mathcal{O}_M(Q)}\|_{Q_i} \\ & \quad \otimes \otimes_{1 \leq i < j \leq q} (\underline{\mathcal{O}_M(Q_i)}\|_{Q_j})^{\otimes -1}. \end{aligned}$$

Thirdly, by (F4), the Serre isometry, we see that the last combination is isometric to

$$\begin{aligned} & \left(\lambda(\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)), h_{F;\omega}(\underline{\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)}) \right) \\ & \quad \otimes \otimes_{i=1}^q \underline{K_M}\|_{Q_i} \otimes \otimes_{i=1}^q \underline{\mathcal{O}_M(Q)}\|_{Q_i} \\ & \quad \otimes \otimes_{1 \leq i < j \leq q} (\underline{\mathcal{O}_M(Q_i)}\|_{Q_j})^{\otimes -1}. \end{aligned}$$

Thus, we get the isometry

$$\begin{aligned} & \left(\lambda(K_M), h_{F;\omega}(\underline{K_M}) \right) \\ & \simeq \left(\lambda(\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)), h_{F;\omega}(\underline{\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)}) \right) \\ & \quad \otimes \otimes_{i=1}^q \underline{K_M}\|_{Q_i} \otimes \otimes_{i=1}^q \underline{\mathcal{O}_M(Q)}\|_{Q_i} \\ & \quad \otimes \otimes_{1 \leq i < j \leq q} (\underline{\mathcal{O}_M(Q_i)}\|_{Q_j})^{\otimes -1}. \end{aligned}$$

Hence, finally, by Remark 4.1.1 and Lemma 4.2.1, we have the isometry

$$\begin{aligned} & \left(\lambda(\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)), h_{F;\omega}(\underline{\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)}_{\omega}) \right) \\ & \simeq \left(\lambda(\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)), h_{F;\omega_{\text{can}}}(\underline{\mathcal{O}_M(Q_1 + \cdots + Q_q - Q)}_{\omega_{\text{can}}}) \right) \end{aligned}$$

which, by definition, completes the proof of the lemma.

Remark 4.2.1. We sometimes call Lemma 4.2.1 and Proposition 4.2.2 Mean Value Lemmas too.

(4.3) With the above definition of ω -Faltings metric, we also have the Noether isometry without any further difficulty. Following Faltings [Fa] and Moret-Bailly [MB], with arithmetic applications in mind, this is done as follows. (I include this subsection simply for completeness which in turn offers me a chance to give one of the main result of this paper, the ω -Noether isometry. If the reader does not want to waste his time on the known discussion about theta norms, he may simply jump to Theorem 4.3.1.)

On the stack \mathcal{U}_q of principally polarized abelian varieties of dimension q , define a degree 2^{2q} covering \mathcal{Q} which classifies pairs consisting of an abelian variety together with a symmetric ample divisor which defines a principal polarization. Similarly, on the stack \mathcal{M}_q of regular algebraic curves of genus q , define the covering \mathcal{P} , which classifies pairs consisting of regular genus q curves together with one of its theta-characters. Then over $\mathbb{Z}[1/2]$, by using the Abel-Jacobi map, there exists a cartesian diagram

$$\begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{Q} \\ \downarrow & & \downarrow \\ \mathcal{M}_q & \rightarrow & \mathcal{U}_q. \end{array}$$

Over \mathcal{Q} , there is the universal abelian variety $p : \mathcal{A} \rightarrow \mathcal{Q}$ together with the theta divisor $\Theta \subset \mathcal{A}$, flat over \mathcal{Q} . Denote the zero section of $\mathcal{A} \rightarrow \mathcal{Q}$ by s , then (up to a universal constant, over the corresponding analytic space,) we have a natural isometry

$$s^*(\underline{\Omega_p^q}) \simeq s^*\underline{\mathcal{O}_{\mathcal{A}}(\Theta)}^{\otimes 2},$$

defined by multiplying θ^{-2} . Here $\underline{\Omega_p^q}$ denotes the line bundle Ω_p^q together with the metric induced from the natural pairing $(\phi, \psi) \mapsto (-1)^{\frac{q(q-1)}{2}} \cdot (\frac{\sqrt{-1}}{2})^q \int_{\mathcal{A}} \phi \wedge \bar{\psi}$, while $\underline{\mathcal{O}_{\mathcal{A}}(\Theta)}$ denotes the line bundle $\mathcal{O}_{\mathcal{A}}(\Theta)$ together with the Hermitian metric defined by using the theta norm, i.e.,

$$\|\theta(Z, z)\|^2 := \sqrt{\det Y} \cdot \exp(-2\pi^t y Y^{-1} y) \cdot |\theta(Z, z)|^2$$

with

$$\theta(Z, z) := \sum_{n \in \mathbb{Z}^q} e^{\pi \sqrt{-1}^t n Z n + 2\pi \sqrt{-1}^t n \cdot z}.$$

Here a principally polarized abelian variety is taking of the form $\mathbb{C}^q / (\mathbb{Z}^q + Z \cdot \mathbb{Z}^q)$ for some complex $q \times q$ matrix Z with positive definite imaginary part Y .

Thus if we denote the universal curve over \mathcal{P} by $p : \mathcal{X} \rightarrow \mathcal{P}$, there is a universal theta-character \mathcal{L} on \mathcal{X} . In particular, the ω -Faltings metric on $\lambda(\mathcal{L})$ gives a Hermitian metric on the associated line bundle over the analytic space corresponding to \mathcal{P} . On the other hand, we know that $\lambda(\mathcal{L})$ is simply the pull-back of $s^*\mathcal{O}(-\Theta)$. So by definition, we see that such an isomorphism gives an isometry for the corresponding Hermitian line bundles if we multiplies the Hermitian norm by $\exp(\delta(M; \omega)/8)$ up to a constant depending only on the genus q on each connected component of \mathcal{P} .

Finally, by (F0), we see that the Hermitian line bundle $\lambda(K_p)$ together with ω -Faltings metric is merely the pull-back of the Hermitian line bundle $s^*\underline{\Omega_p^q}$. Thus,

by using Proposition 4.2.2, together with the detailed discussion of the constants appeared above by Moret-Bailly in [MB], we arrive at the following

Theorem 4.3.1. (ω -Noether isometry) *With respect to the normalized volume ω (associated to a quasi-hyperbolic metric) on a compact Riemann surface M , for any ω -admissible Hermitian line bundle \bar{L} , we have the following isometry:*

$$\left(\lambda(L), h_{F;\omega}(\bar{L})\right)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \underline{K_M}, \underline{K_M} \rangle \otimes \mathcal{O}(e^{\delta(M)} \cdot (2\pi)^{-4q}). \quad (4.3.1)$$

Remark 4.3.1. The discussion in this section works is simply due to the fact that we here use only the normalized ω -Arakelov metrics on K_M and $\mathcal{O}_M(P)$.

Remark 4.3.2. Note that the Riemann-Roch isometry gives the difference between Hermitian line bundles $\left(\lambda(K_M^{\otimes n}), h_{F;\omega}(\underline{K_M}^{\otimes n})\right)$ and $\left(\lambda(K_M), h_{F;\omega}(\underline{K_M})\right)$ for all half integers n . So there are still some freedom for us to choose the ω -Riemann-Roch metric. But the above discussion gives summation of $\left(\lambda(K_M^{\otimes 1/2}), h_{F;\omega}(\underline{K_M}^{\otimes 1/2})\right)$ and $\left(\lambda(K_M), h_{F;\omega}(\underline{K_M})\right)$ for ω -Faltings metrics. So we get a unique metric $h_{F;\omega}(\bar{L})$ for all ω -admissible Hermitian line bundles \bar{L} , so as to obtaining the Noether isometry. Such an idea was also previously used by Beilinson and Manin in [BM] when they gave Mumford volume forms on moduli spaces of compact Riemann surfaces. It is for this reason we call (3.2.1) the weak Riemann-Roch theorem, while we call the Deligne-Riemann-Roch theorem the strong Riemann-Roch theorem. (See e.g., [We1, A1].)

§5. New metrics on determinant of cohomology for singular metrics and Mumford type isometries

(5.1) We start with the following

Theorem 5.1.1. (Deligne-Riemann-Roch Isometry for Singular Metrics) *For any normalized volume form ω on a compact Riemann surface M associated to a smooth metric or a quasi-hyperbolic metric, for any ω -admissible metric \bar{L} on M , there exists an ω -determinant metric $h_{\overline{K_M}}(\bar{L})$ on $\lambda(L)$ such that we have the following canonical isometry*

$$\left(\lambda(L), h_{\overline{K_M}}(\bar{L})\right)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \overline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \overline{K_M}, \overline{K_M} \rangle \otimes \mathcal{O}(e^{a(q)}). \quad (5.1.1)$$

Here $a(q) := (1 - q)(24\zeta'_{\mathbb{Q}}(-1) - 1)$ denotes the Deligne constant.

Proof. First, let us prove the theorem when the metric on K_M is simply the ω -Arakelov metric, i.e., we for the time being assume that $\overline{K_M} = \underline{K_M}$. In this case, by Theorem 4.3.1, for any ω -admissible metric \bar{L} on M , there exists a metric $h_{F,\omega}(\bar{L})$ on $\lambda(L)$ such that we have the following ω -Noether isometry:

$$\left(\lambda(L), h_{F,\omega}(\bar{L})\right)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \underline{K_M}, \underline{K_M} \rangle \otimes \mathcal{O}(e^{\delta(M)} \cdot (2\pi)^{-4q}). \quad (5.1.2)$$

As a direct consequence, if we set

$$h_{\underline{K_M}}(\bar{L}) := h_{F,\omega}(\bar{L}) \cdot e^{-\delta(M,\omega)/12} \cdot (2\pi)^{4q/12} \cdot e^{a(q)/12}, \quad (5.1.3)$$

with $\delta(\omega, M) = \delta(M)$ the Faltings δ -function, we then arrive at the isometry

$$\left(\lambda(L), h_{\underline{K_M}}(\bar{L})\right)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \underline{K_M}, \underline{K_M} \rangle \otimes \mathcal{O}(e^{a(q)}). \quad (5.1.4)$$

In general, we have $\overline{K_M} = \underline{K_M} \cdot \mathcal{O}_M(e^c)$ for a certain constant function c on M by applying the ω -admissible condition. So up to a constant $A(c, q, d)$, which can be easily evaluated by using ω -intersection and depends only on c and q and the degree d of L ,

$$\langle \bar{L}, \bar{L} \otimes \underline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \underline{K_M}, \underline{K_M} \rangle$$

is simply

$$\langle \bar{L}, \bar{L} \otimes \overline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \overline{K_M}, \overline{K_M} \rangle.$$

(We leave the precise valuation of $A(c, q, d)$ to the reader as it is an interesting exercise to understand the Polyakov variation formula for our metric.) Set

$$h_{\overline{K_M}}(\bar{L}) := h_{\underline{K_M}}(\bar{L}) \cdot A(c, q, d),$$

then we have the Deligne-Riemann-Roch isometry stated in the theorem. This completes the proof of the theorem.

As direct consequences, we have the following.

Corollary 5.1.2. (Mean Value Lemma) *With the same notation as above, suppose that ω_1 and ω_2 are two normalized volume forms on M , then there exists a canonical isometry*

$$\left(\lambda(K_M), h_{\underline{K_M}_{\omega_1}}(\underline{K_M}_{\omega_1})\right) \simeq \left(\lambda(K_M), h_{\underline{K_M}_{\omega_2}}(\underline{K_M}_{\omega_2})\right).$$

Proof. By applying Theorem 5.1.1, the isometry is obtained by the Mean Value Lemma in ω -intersection theory, i.e., Proposition 2.5.3.

For completeness, we give the following better than nothing

Corollary 5.1.3. *With the same notation as above, assume that ω is smooth on M , then $h_{\overline{K_M}}(\bar{L})$ is the Quillen metric $h_Q(\overline{K_M}, \bar{L})$ on $\lambda(L)$ associated to the metric on M induced from $\overline{K_M}$ and the metrized line bundle \bar{L} .*

Proof. By applying Deligne-Riemann-Roch theorem for Quillen metric $h_Q(\overline{K_M}, \bar{L})$, (see e.g. [De2] together with [So],) we have the isometry

$$\left(\lambda(L), h_Q(\overline{K_M}, \bar{L})\right)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \overline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \overline{K_M}, \overline{K_M} \rangle \otimes \mathcal{O}(e^{a(q)}). \quad (5.1.2)$$

Comparing this with (5.1.4), we complete the proof of this corollary.

Moreover, from the above, easily, one sees that the determinant metric $h_{\overline{K_M}}(\bar{L})$ introduced in Theorem 5.1.1 is compactible with the normalization process given in §3. That is to say, we have the Polyakov variation formula, the Mean Value Lemmas, among others. Therefore, all the above discussion is compactible. Surely, $h_{\overline{K_M}}(\bar{L})$ is then the new metric on $\lambda(L)$ associated to possibly singular metrics on $\overline{K_M}$ and on \bar{L} we seek at the very beginning. In the sequel, we give some applications and a more geometric interpretation of this new metric.

(5.2) We start with a more suitable version of Deligne-Riemann-Roch isomorphism for punctured Riemann surfaces.

First, the algebraic Noether theorem tells us that for a compact Riemann surface M there is a canonical isomorphism

$$\lambda(\mathcal{O}_M)^{\otimes 12} \simeq \langle K_M, K_M \rangle.$$

Secondly, the adjunction isomorphism gives the following isomorphisms

$$\langle K_M(P_i), P_i \rangle \simeq \mathcal{O}.$$

Here P_i , $i = 1, \dots, N$, denotes the punctures of M . As a direct consequence, we

have

$$\begin{aligned}
& \langle K_M, K_M \rangle \\
& \simeq \langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle \\
& \quad \otimes \left(\langle K_M, \mathcal{O}_M(P_1 + \cdots + P_N) \rangle \otimes \langle \mathcal{O}_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle \right)^{-1} \\
& \simeq \langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle \\
& \quad \otimes \left(\langle K_M, \mathcal{O}_M(P_1 + \cdots + P_N) \rangle \otimes \bigotimes_{i=1}^N \langle \mathcal{O}_M(P_i), K_M(P_i) \otimes \bigotimes_{j=1, \neq i}^N \mathcal{O}_M(P_j) \rangle \right)^{-1} \\
& \simeq \langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle \\
& \quad \otimes \left(\langle K_M, \mathcal{O}_M(P_1 + \cdots + P_N) \rangle \otimes \bigotimes_{i=1}^N \langle \mathcal{O}_M(P_i), \bigotimes_{j=1, \neq i}^N \mathcal{O}_M(P_j) \rangle \right)^{-1} \\
& \simeq \langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle \\
& \quad \otimes \left(\langle K_M, \mathcal{O}_M(P_1 + \cdots + P_N) \rangle \otimes \bigotimes_{1 \leq i < j \leq N} \langle \mathcal{O}_M(P_i), \mathcal{O}_M(P_j) \rangle^{\otimes 2} \right)^{-1}.
\end{aligned}$$

Thus if we set

$$\Delta_\alpha = \begin{cases} \langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle, & \text{if } \alpha = 0, \\ \bigotimes_{k=1}^N \langle \mathcal{O}_M(P_k), \mathcal{O}_M(P_k) \rangle (= \bigotimes_{k=1}^N \langle K_M, \mathcal{O}_M(P_k) \rangle^{\otimes -1}), & \text{if } \alpha = 1, \\ \bigotimes_{1 \leq i < j \leq N} \langle \mathcal{O}_M(P_i), \mathcal{O}_M(P_j) \rangle, & \text{if } \alpha = 2. \end{cases} \quad (5.2.1)$$

we have the following canonical isomorphism

$$\lambda(\mathcal{O}_M)^{\otimes 12} \simeq \Delta_0 \otimes \Delta_1 \otimes \Delta_2^{\otimes -2}. \quad (5.2.2)$$

In this way, we arrive at the following version of Deligne-Riemann-Roch isomorphism, which is the most suitable one for punctured Riemann surfaces.

Theorem 5.2.1. (Deligne-Riemann-Roch Theorem) *With the same notation as above, for all line bundles L on M , we have*

$$\lambda(L)^{\otimes 12} \simeq \Delta_0 \otimes \Delta_1 \otimes \Delta_2^{\otimes -2} \otimes \langle L, L \otimes K_M^{\otimes -1} \rangle^{\otimes 6}. \quad (5.2.3)$$

Remark 5.2.1. The reader may wonder why we use Δ_0 as a very basic object to build up the isomorphism. We here justify our choice by the following two reasons: first of all, the logarithmic geometry says that for punctured Riemann surfaces, $K_M(P_1 + \cdots + P_N)$ is far more natural (see e.g., [Fu]); second, we will see that on the Teichmüller space of the punctured Riemann surfaces of signature (q, N) , the most natural line bundle corresponding to the Weil-Petersson Kähler form is given by the Deligne pairing $\langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle$, rather than $\langle K_M, K_M \rangle$.

To go further, for punctures Riemann surfaces M^0 with cusps P_1, \dots, P_N and the smooth compactification M , we define the Mumford type line bundles λ_n by

$$\lambda_n := \begin{cases} \lambda(K_M^{\otimes n} \otimes (\mathcal{O}_M(P_1 + \dots + P_N))^{\otimes n-1}), & \text{if } n > 0; \\ \lambda(\mathcal{O}_M), & \text{if } n = 0; \\ \lambda((K_M(P_1 + \dots + P_N))^{\otimes n}), & \text{if } n < 0. \end{cases} \quad (5.2.4)$$

Remark 5.2.2. For the time being, we justify this definition of Mumford type line bundles λ_n as follows. First of all, the most natural line bundle associated to a punctured Riemann surface is the associated logarithmic tangent line bundle, so it is fairly natural to define λ_n for n negative by setting

$$\lambda_n := \lambda((K_M(P_1 + \dots + P_N))^{\otimes n}).$$

Second, Serre duality should give intrinsic relations among all λ_n 's. This then gives the above definition of λ_n for n positive.

Theorem 5.2.2. (Generalized Mumford Relations) *With the same notation as above, for all positive integers n , we have the following isomorphisms:*

- (a) $\lambda_n \simeq \lambda_{1-n}$;
- (b) $\lambda_n^{\otimes 12} \simeq \Delta_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1 \otimes \Delta_2^{\otimes 10-12n}$, and
- (c) $\lambda_n \simeq \lambda_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1^{\otimes -\frac{n(n-1)}{2}} \otimes \Delta_2^{\otimes(n-1)^2}$.

Proof. (a) is a direct consequence of the definition. The proofs of (b) and (c) are similar, so we only give the one for (b). For this latest purpose, we have the following calculation. In Theorem 5.2.1, setting $L := K_M^{\otimes n} \otimes (\mathcal{O}_M(P_1 + \dots + P_N))^{\otimes n-1}$, we get

$$\begin{aligned} & \lambda_n^{\otimes 12} \\ & \simeq \Delta_0 \otimes \Delta_1 \otimes \Delta_2^{\otimes -2} \otimes \langle K_M^{\otimes n} \otimes (\mathcal{O}_M(P_1 + \dots + P_N))^{\otimes n-1}, \\ & \quad K_M^{\otimes n} \otimes (\mathcal{O}_M(P_1 + \dots + P_N))^{\otimes n-1} \otimes K_M^{\otimes -1} \rangle^{\otimes 6} \\ & \simeq \Delta_0 \otimes \Delta_1 \otimes \Delta_2^{\otimes -2} \\ & \quad \otimes \langle K_M^{\otimes n} \otimes (\mathcal{O}_M(P_1 + \dots + P_N))^{\otimes n-1}, K_M(P_1 + \dots + P_N) \rangle^{\otimes 6(n-1)} \\ & \simeq \Delta_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1 \otimes \Delta_2^{\otimes -2} \\ & \quad \otimes \langle \mathcal{O}_M(P_1 + \dots + P_N), K_M(P_1 + \dots + P_N) \rangle^{\otimes -6(n-1)}. \end{aligned}$$

Hence, by the adjunction isomorphism, as in the proof of (5.2.2), we have

$$\lambda_n^{\otimes 12} \simeq \Delta_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1 \otimes \Delta_2^{\otimes -12n+10}.$$

This completes the proof of the theorem.

(5.3) Now we give the counter part of the metric theory for the discussion in (5.2). We start with a discussion on results in which only ω -Arakelov metrics on both the canonical line bundle and pointed line bundles associated to cusps are used.

For a normalized volume form ω on M , define the following metrized lines:

$$\underline{\lambda}_n := \begin{cases} \left(\lambda_n, h_{\underline{K}_M}(\underline{K}_M^{\otimes n} \otimes \mathcal{O}_M(\underline{P}_1 + \cdots + \underline{P}_N)^{\otimes n-1}) \right), & \text{if } n > 0; \\ \left(\lambda_0, h_{\underline{K}_M}(\underline{\mathcal{O}}_M) \right), & \text{if } n = 0; \\ \left(\lambda_n, h_{\underline{K}_M}((\underline{K}_M(\underline{P}_1 + \cdots + \underline{P}_N))^{\otimes n}) \right), & \text{if } n < 0. \end{cases} \quad (5.3.1)$$

$$\underline{\Delta}_n := \begin{cases} \langle \underline{K}_M(\underline{P}_1 + \cdots + \underline{P}_N), \underline{K}_M(\underline{P}_1 + \cdots + \underline{P}_N) \rangle, & \text{if } n = 0; \\ \otimes_{k=1}^N \langle \underline{\mathcal{O}}_M(\underline{P}_k), \underline{\mathcal{O}}_M(\underline{P}_k) \rangle, & \text{if } n = 1; \\ \otimes_{1 \leq i < j \leq N} \langle \underline{\mathcal{O}}_M(\underline{P}_i), \underline{\mathcal{O}}_M(\underline{P}_j) \rangle, & \text{if } n = 2. \end{cases} \quad (5.3.2)$$

Note that in this case the adjunction isometry holds, so the proof for the algebraic results in (5.2) is valid here without any essential change. In other words, we have the following

Theorem 5.3.1. *With the same notation as above, for any positive integer n , we have the following isometries:*

(a) (Serre isometry)

$$\underline{\lambda}_n \simeq \underline{\lambda}_{1-n};$$

(b) (Generalized Mumford isometry)

$$\underline{\lambda}_n^{\otimes 12} \simeq \underline{\Delta}_0^{\otimes 6n^2 - 6n + 1} \otimes \underline{\Delta}_1 \otimes \underline{\Delta}_2^{\otimes -12 + 10} \otimes \mathcal{O}(e^{a(q)});$$

(c) (Generalized Mumford isometry)

$$\underline{\lambda}_n \simeq \underline{\lambda}_1^{\otimes 6n^2 - 6n + 1} \otimes \underline{\Delta}_1^{\otimes -\frac{n(n-1)}{2}} \otimes \underline{\Delta}_2^{\otimes (n-1)^2} \otimes \mathcal{O}(e^{-\frac{n(n-1)}{2} \cdot a(q)}).$$

Proof. As said before, the proof of this theorem is essentially given in (5.2), as we here use the ω -Arakelov metrics so that the adjunction isometry holds. With this, by using Theorem 5.1.1 with a tedious calculation, we complete the proof of this theorem.

(5.4) More generally, without using the adjunction isometry and with the application to the moduli problems in mind, we in this subsection give a generalization of Theorem 5.3.1. As in (5.3), we always fix a normalized volume form ω on M .

For an $n+1$ -tuple of real numbers $(\alpha; \beta_1, \dots, \beta_N)$, define the associated metrized lines as follows:

$$\overline{\lambda}_n^{\alpha; \beta} := \begin{cases} \left(\lambda_n, h_{\overline{K}_M}^\alpha((\overline{K}_M)^\alpha)^{\otimes n} \otimes \mathcal{O}_M(\overline{P}_1^{\beta_1} + \dots + \overline{P}_N^{\beta_N})^{\otimes n-1} \right), & \text{if } n > 0; \\ \left(\lambda_0, h_{\overline{K}_M}^\alpha(\mathcal{O}_M) \right), & \text{if } n = 0; \\ \left(\lambda_n, h_{\overline{K}_M}^\alpha((\overline{K}_M)^\alpha(\overline{P}_1^{\beta_1} + \dots + \overline{P}_N^{\beta_N}))^{\otimes n} \right), & \text{if } n < 0; \end{cases} \quad (5.4.1)$$

and

$$\overline{\Delta}_n^{\alpha; \beta} := \begin{cases} \langle \overline{K}_M^\alpha(\overline{P}_1^{\beta_1} + \dots + \overline{P}_N^{\beta_N}), \overline{K}_M^\alpha(\overline{P}_1^{\beta_1} + \dots + \overline{P}_N^{\beta_N}) \rangle, & \text{if } n = 0; \\ \langle \overline{K}_M^\alpha, \mathcal{O}_M(\overline{P}_1^{\beta_1} + \dots + \overline{P}_N^{\beta_N})^{\otimes -1} \rangle, & \text{if } n = 1; \\ \langle \overline{K}_M^\alpha(\overline{P}_1^{\beta_1} + \dots + \overline{P}_N^{\beta_N}), \mathcal{O}_M(\overline{P}_1^{\beta_1} + \dots + \overline{P}_N^{\beta_N})^{\otimes \frac{1}{2}} \rangle, & \text{if } n = 2. \end{cases} \quad (5.4.2)$$

Here $\overline{K}_M^\alpha := \underline{K}_M \otimes \mathcal{O}_M(e^\alpha)$ and $\overline{P}_i^{\beta_i} := \underline{P}_i \otimes \mathcal{O}_M(e^{\beta_i})$, $i = 1, \dots, N$. Then we get the following

Theorem 5.4.1. *With the same notation as above, for any positive integer n , we have the following isometries:*

(a) (Serre isometry)

$$\overline{\lambda}_n^{\alpha; \beta} \simeq \overline{\lambda}_{1-n}^{\alpha; \beta};$$

(b) (Generalized Mumford isometry)

$$(\overline{\lambda}_n^{\alpha; \beta})^{\otimes 12} \simeq (\overline{\Delta}_0^{\alpha; \beta})^{\otimes 6n^2 - 6n + 1} \otimes (\overline{\Delta}_1^{\alpha; \beta}) \otimes (\overline{\Delta}_2^{\alpha; \beta})^{\otimes -12 + 10} \otimes \mathcal{O}(e^{a(q)});$$

(c) (Generalized Mumford isometry)

$$\overline{\lambda}_n^{\alpha; \beta} \simeq (\overline{\lambda}_1^{\alpha; \beta})^{\otimes 6n^2 - 6n + 1} \otimes (\overline{\Delta}_1^{\alpha; \beta})^{\otimes -\frac{n(n-1)}{2}} \otimes (\overline{\Delta}_2^{\alpha; \beta})^{\otimes (n-1)^2} \otimes \mathcal{O}(e^{-\frac{n(n-1)}{2} \cdot a(q)}).$$

Proof. By Proposition 3.2.3, i.e., the Serre duality, we have (a). The proofs of (b) and (c) come directly from applying the Deligne-Riemann-Roch isometry for singular metrics, i.e., Theorem 5.1.1. Indeed, we, in the above definition (5.4.2) on $\overline{\Delta}_1^{\alpha; \beta}$ and $\overline{\Delta}_2^{\alpha; \beta}$, already made a subtle change from these in (5.3.2). Moreover, it

is clear by a direct calculation that with (5.4.2) no adjunction isometry is needed to get (b) and (c). We leave the details to the reader.

Appendix to §5. Universal Riemann-Roch Isomorphism

Even though the above discussion is about a single punctured Riemann surface, but it is clear that it works for a family. For our own convenience (to the later discussion) and for the completeness, we in this appendix to §5 study briefly the structure of the moduli space of punctured Riemann surfaces and their compactifications, which is well-known to experts. The details for most of the statements, except for the universal Riemann-Roch theorem for punctured curves, may be found in [DM] and Kundsén's series of papers ([KM] and [Kn]).

Denote the moduli space of N ordered pointed compact Riemann surfaces of genus q by $\mathcal{M}_{q,N}$, and its universal curve by $\pi_{q,N} : \mathcal{C}_{q,N} \rightarrow \mathcal{M}_{q,N}$. (Here we for simplicity assume that $\mathcal{M}_{q,N}$ is a fine moduli. If the reader does not like this, she or he should then use V -manifold language or algebraic stack language to justify what follows.)

Naturally, we have the relative canonical line bundle $K_{q,N}$ for $\pi_{q,N}$, and N -sections $\mathbb{P}_1, \dots, \mathbb{P}_N$ corresponding to N points on each surface. One knows that $\mathcal{C}_{q,N}$ may be viewed as $\mathcal{M}_{q,N+1}$, while $\pi_{q,N}$ is simply the map of dropping the last, i.e., the $(N+1)$ -th point. In particular, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}_{q,N+1} & = & \mathcal{C}_{q,N} & \xrightarrow{\phi_{q,N}} & \mathcal{C}_{q,N-1} & = & \mathcal{M}_{q,N} \\ & & \pi_{q,N} \downarrow & & \downarrow \pi_{q,N-1} & & \\ & & \mathcal{M}_{q,N} & \xrightarrow{\pi_{q,N-1}} & \mathcal{M}_{q,N-1} & & . \end{array}$$

Here $\phi_{q,N}$ viewed as a morphism from $\mathcal{M}_{q,N+1}$ to $\mathcal{M}_{q,N}$ is simply the morphism defined by dropping the second to the last point.

To compactify $\mathcal{M}_{q,N}$, we need to add two types of boundaries. That is, the boundaries coming from the degeneration of compact Riemann surfaces of genus q , and the boundaries coming from the degeneration of punctures. For our own convenience, we call the first type of boundaries the absolute horizontal boundaries, while we call the second type of boundaries the relative horizontal boundaries.

For absolute horizontal boundaries, it is well-known that as a codimension one subvariety, it consists of $[\frac{g}{2}] + 1$ irreducible components $\Delta_0, \Delta_1, \dots, \Delta_{[\frac{g}{2}]}$. (Please do confuse Δ here with Δ elsewhere in this paper.) Indeed, such boundaries may

be at best understood via the universal curve: for a general point $x \in \Delta_0$, the corresponding fiber is a genus q curve with one non-separating node; while for a general point $x \in \Delta_i$, $i = 1, \dots, [\frac{q}{2}]$, the corresponding fiber is a genus q curve with one separating node whose only two irreducible components are smooth and of genera i and $q - i$ respectively.

To understand the relative horizontal boundaries, we suggest the reader to consult papers on resolution of diagonals, say, [BG] and [FM]. In any case, this may be understood by considering what happens for a general compact Riemann surface. So we for the time being assume that M is a compact Riemann surface without non-trivial automorphisms. Then the moduli $M^{(1)}$ of a point on M is simply M itself. And the universal curve over $M^{(1)}$ is simply $M \times M$ with the section given by the diagonal. $M \times M$ can be viewed as the moduli $M^{(2)}$ of an ordered pairing on M . There is no problem to see the fiber together with two sections of the universal curve over points in $M^{(2)}$ which are away from the diagonal: the fiber is simply M itself together with two distinct points (as we assume that the base point in $M^{(2)}$ is off diagonal). On the other hand, if the point in $M^{(2)}$ is on the diagonal, two sections intersect each other. So we cannot simply find two distinct points on M , the fake fiber in the universal curve. To remedy this, Grothendieck-Mumford-Knudsen first blow up M at this point so that two points can be pulled apart. In other words, the fibers over a point on the diagonal now admits two irreducible components: one is the original curve, while the other is a projective line together with three marked points – the intersection point with M representing the center of the blowing up, while the other two points representing two infinitesimal points over the intersection of M with \mathbb{P}^1 . In this way, in particular, we see that the universal curve admits two sections which can never meet each other.

This picture may be generalized to the moduli $M^{(N)}$ of ordered N points for M . To describe it together with its universal curve, we consider the set $\{1, \dots, N\}$. For each subset S of $\{1, \dots, N\}$ with cardinal number $\#S$ at least two, we have an S -diagonal D_S in M^N . We know that to have $M^{(N)}$, we need to blow up these diagonals, so that we then get normal crossing divisors Δ_S resulting from these diagonals D_S (and exceptional divisors). In particular, for a general point $x \in \Delta_S$, the fiber of the universal curve consists of two irreducible components, one is the original curve M while the other is the projective line \mathbb{P}^1 . Moreover, on M there are $N - \#S$ marks and on \mathbb{P}^1 , there are remaining $\#S$ -marks. Similarly, as for the case when $N = 2$, now on the universal curve, there are N -sections which do not

intersect pairwise.

From the above discussion, we see that the absolute horizontal boundaries consist of divisors $\Delta_0, \Delta_1, \dots, \Delta_{[\frac{q}{2}]}$, and the relative horizontal boundaries consist of divisors Δ_S for each subset S of $\{1, \dots, N\}$ with cardinal number at least two.

Universal Riemann-Roch Theorem. *With the same notation as above, denote by $\bar{\pi}_{q,N} : \overline{\mathcal{C}}_{q,N} \rightarrow \overline{\mathcal{M}}_{q,N}$ the universal curve over stably compactified moduli space of punctured Riemann surfaces of signature (q, N) , and by K the associated relative canonical line bundle, then for any line bundle L on $\overline{\mathcal{C}}_{q,N}$, we have the canonical isomorphism*

$$\lambda(L)^{\otimes 12} \simeq \langle L, L \otimes K \rangle^{\otimes 6} \otimes \langle K, K \rangle \otimes \bigotimes_{i=0}^{[\frac{q}{2}]} \mathcal{O}(\Delta_i) \otimes \bigotimes_{S \subset \{1, \dots, N\}, \#S \geq 2} \mathcal{O}(\Delta_S).$$

The proof may be given by using Mumford's argument on Riemann-Roch theorem for the universal curve over stably moduli space of compact Riemann surfaces, as only ordinary double points are involved here. We leave this to the reader. (See however [We2].)

To end this appendix to section 5, we give the following list of relations for line bundles associated to $\bar{\pi}_{q,N}$ coming from the intersection.

- (a) $\langle \mathbb{P}_i, \mathbb{P}_j \rangle \simeq \mathcal{O}$, if $i, j = 1, \dots, N$ and $i \neq j$;
- (b) $\langle K(\mathbb{P}_i), \mathbb{P}_i \rangle \simeq \mathcal{O}$, if $i = 1, \dots, N$;
- (c) $\langle K, \mathbb{P}_i \rangle \simeq \left(\pi_{q,N-1}^* \langle K, \mathbb{P}_i \rangle \right) (\mathbb{P}_i)$, if $i = 1, \dots, N-1$;
- (d) $\langle K, \mathbb{P}_N \rangle \simeq K(\mathbb{P}_1 + \dots + \mathbb{P}_{N-1})$;
- (e) $K(\mathbb{P}_1 + \dots + \mathbb{P}_N) = \phi_{q,N}^* \left(K(\mathbb{P}_1 + \dots + \mathbb{P}_{N-1}) \right)$.

Indeed, with these relations, we may easily generate many interesting relations such as the dilation equation and the string equation. We leave all this to the reader.

§6. Arakelov-Poincaré volume and a geometric interpretation of our new metrics

(6.1) From now on, we will apply our admissible theory to singular hyperbolic metrics. For doing so, we need to understand how the geometrically defined hyperbolic

metric on the logarithmic tangent line bundle relates with the arithmetically defined hyperbolic-Arakelov metric on the canonical line bundle. We bridge them via an invariant, the Arakelov-Poincaré volume, for a punctured Riemann surface. We end this section with a geometric interpretation for our new metric on determinant of cohomology.

Let us start with a discussion on hyperbolic metrics on punctured Riemann surfaces. As before, denote by ω_{hyp} the normalized volume form associated to the standard hyperbolic metric τ_{hyp}^0 on a punctured Riemann surface M^0 of signature (q, N) . Thus, in particular, if we denote the corresponding volume form (with respect to τ_{hyp}^0) by $d\mu_{\text{hyp}}$, then $\int_{M^0} d\mu_{\text{hyp}} = 2\pi(2q-2+N)$, and $2\pi(2q-2+N)\omega_{\text{hyp}} = d\mu_{\text{hyp}}$.

For τ_{hyp}^0 , or equivalently for $d\mu_{\text{hyp}}$ on M^0 , if we view them as a singular metric on M , the smooth compactification of M^0 , then the natural line bundle we should attach to it is the so-called logarithmic tangent bundle $T_M\langle\log D\rangle$. Here D denotes the divisor at infinity, or the cuspidal divisor, i.e., $P_1 + \cdots + P_N$. (See e.g., [De1], [Mu1] or [Fu]). Over the compact Riemann surface M , $T_M\langle\log D\rangle$ is nothing but the dual of the line bundle $K_M(P_1 + \cdots + P_N)$. Here as before K_M denotes the canonical line bundle of M . So if we denote the induced Hermitian metric from τ_{hyp}^0 on $K_M(P_1 + \cdots + P_N)$ by $\tau_{\text{hyp};K_M(D)}^\vee$, we get the following Einstein equation

$$c_1\left(K_M(P_1 + \cdots + P_N), \tau_{\text{hyp};K_M(D)}^\vee\right) = d\mu_{\text{hyp}} = (2q - 2 + N)\omega_{\text{hyp}}. \quad (6.1.1)$$

We are not quite satisfied with this, as the metric discussed above only has its nice meaning on the logarithmic tangent bundle. In particular, it does not give us any indication on how to get an ω_{hyp} -admissible metric on K_M , without which we cannot apply our admissible theory. So we should seek new admissible metrics $\rho_{\text{hyp};K_M}$ and $\rho_{\text{hyp};P_i}$ on K_M and on $\mathcal{O}_M(P_i)$, $i = 1, \dots, N$, respectively, which naturally come from the standard hyperbolic metric. More precisely, for the time being, the picture we have in mind for these admissible metrics is that they are very natural in the following sense:

- (i) they should be ω_{hyp} -admissible;
- (ii) they should give the following decomposition of the standard hyperbolic metric on $K_M(P_1 + \cdots + P_N)$

$$\tau_{\text{hyp};K_M(D)}^\vee = \rho_{\text{hyp};K_M} \otimes \rho_{\text{hyp};P_1} \otimes \cdots \otimes \rho_{\text{hyp};P_N}; \quad (6.1.2)$$

(iii) they should obey the residue isometry, i.e.,

$$(K_M(P_i), \rho_{\text{hyp}; K_M} \otimes \rho_{\text{hyp}; P_i})|_{P_i} \simeq \mathbb{C} \quad (6.1.3)$$

for all $i = 1, \dots, N$.

Remark 6.1.1. There is an interesting metric on K_M induced from $d\mu_{\text{hyp}}$, i.e., view it as the dual of the tangent bundle T_M , then the singular volume form $d\mu_{\text{hyp}} = g(z) \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ will give a singular metric via the function g on T_M and hence on K_M . But this metric on K_M is unfortunately not ω_{hyp} -admissible. (Otherwise, the problem should be much easier.)

Before defining the above metrics on K_M and on $\mathcal{O}_M(P_i)$, $i = 1, \dots, N$, respectively, motivated by our previous work on admissible theory for smooth hyperbolic metrics in [We1], we now introduce an invariant $A_{\text{Ar, hyp}}(M^0)$, the Arakelov-Poincaré volume, associated to a punctured Riemann surface M^0 as follows.

First of all, following Selberg, define the so-called Selberg zeta function $Z_{M^0}(s)$ of M^0 for $\text{Re}(s) > 1$ by the absolutely convergent product

$$Z_{M^0}(s) := \prod_{\{l\}} \prod_{m=0}^{\infty} (1 - e^{-(s+m)|l|}), \quad (6.1.4)$$

where l runs over the set of all simple closed geodesics on M^0 with respect to the hyperbolic metric $d\mu_{\text{hyp}}$ on M^0 , and $|l|$ denotes the length of l . It is known that by using Selberg trace formula for weight zero forms the function $Z_{M^0}(s)$ admits a meromorphic continuation to the whole complex s -plane which has a simple zero at $s = 1$. Secondly, motivated by the work of D'Hoker-Phong ([D'HP]) and Sarnak ([Sa]), we introduce the following factorization for the Selberg zeta function:

$$Z_{M^0}(s) =: \det(\Delta_{\text{hyp}} + s(s-1)) \cdot \mathbb{N}(s)^{2q-2+N}. \quad (6.1.5)$$

Here Δ_{hyp} denotes the hyperbolic Laplacian on M^0 , $\mathbb{N}(s)$ denotes the function

$$\mathbb{N}(s) := \frac{e^{-E+s(s-1)}}{2\pi^s} \cdot \frac{\Gamma(s)}{(\Gamma_2(s))^2} \quad (6.1.6)$$

with $E = -\frac{1}{4} - \frac{1}{2} \log 2\pi + 2\zeta'_{\mathbb{Q}}(-1)$, $\Gamma(s)$ the ordinary gamma function, and $\Gamma_2(s)$ the Barnes double gamma function. Thirdly, define the regularized determinant for the Laplacian Δ_{hyp} by

$$\det^*(\Delta_{\text{hyp}}) := \frac{d}{ds} \left(\det(\Delta_{\text{hyp}} + s(s-1)) \right) \Big|_{s=1}. \quad (6.1.7)$$

Finally, define the *Arakelov-Poincaré volume* $A_{\text{Ar,hyp}}(M^0)$ for M^0 via the formula:

$$\log A_{\text{Ar,hyp}}(M^0) := a_{\text{Ar,hyp}} := \frac{12}{2} \cdot \frac{1}{2q-2} \cdot \left(\log \frac{\det^* \Delta_{\text{Ar}}}{A_{\text{Ar}}(M)} - \log \frac{\det^* \Delta_{\text{hyp}}}{2\pi(2q-2)} \right). \quad (6.1.8)$$

Here Δ_{Ar} denotes the Laplacian for the Arakelov metric on M , $A_{\text{Ar}}(M)$ denotes the volume of M with respect to the Arakelov metric.

Remark 6.1.2. By definition, we know that, up to a constant factor depending only on the signature (q, N) of M^0 , $\det^*(\Delta_{\text{hyp}})$ is simply $Z'_{M^0}(1)$. We leave this interesting point to the reader. Please also carefully compare our definition (6.1.7) of the regularized determinant for the Laplacian with the one proposed by Efrat in the one page correction of [Ef].

Remark 6.1.3. Obviously, the Arakelov-Poincaré volume is a very natural invariant for the punctured Riemann surface M^0 , hence can be viewed as a certain interesting function on the Teichmüller space $T_{q,N}$ of punctured Riemann surfaces of signature (q, N) . The reader may consult [We1] for the degeneration behavior of this invariant when $N = 0$.

(6.2) With above discussion on the Arakelov-Poincaré volume for M^0 , we are ready to introduce the geometrically natural admissible metrics on K_M and $\mathcal{O}_M(P_i)$, $i = 1, \dots, N$.

Undoubtedly, the first point is that two ω_{hyp} -admissible metrics on a fixed line bundle differ only by a constant factor. The second point is that we have already had arithmetically natural admissible metrics on K_M and $\mathcal{O}_M(P_i)$, i.e., the corresponding ω_{hyp} -Arakelov metrics $\rho_{\text{Ar}, \omega_{\text{hyp}}}$ and $\rho_{\text{Ar}, \omega_{\text{hyp}}, P_i}$, $i = 1, \dots, N$, respectively. Hence, the geometrically natural admissible metrics on K_M and $\mathcal{O}_M(P_i)$ we seek should be proportional to the corresponding arithmetically natural admissible metrics defined by using hyperbolic Green's functions.

With this in mind, we define the geometrically natural admissible metric on K_M by multiplying the ω_{hyp} -Arakelov metric $\rho_{\text{Ar}, \omega_{\text{hyp}}}$ the factor $A_{\text{Ar,hyp}}(M^0)$. Denote the resulting Hermitian line bundle by $\underline{K}_{M,\text{hyp}}$. That is, we have

$$\underline{K}_{M,\text{hyp}} := \underline{K}_{M,\omega_{\text{hyp}}} \cdot A_{\text{Ar,hyp}}(M^0), \quad (6.2.1)$$

or equivalently,

$$\rho_{\text{hyp}; K_M} := \rho_{\text{Ar}, \omega_{\text{hyp}}} \cdot A_{\text{Ar,hyp}}(M^0). \quad (6.2.2)$$

Once a geometrically meaningful admissible metric is introduced on K_M , we are left with the only problem to define a similar metric on the cuspidal line bundle.

For this purpose, we introduce the following additional principal: for our theory of metrics, all the punctures should be viewed as the same, i.e., there should be no difference when we impose the geometrically meaningful admissible metrics $\rho_{\text{hyp};P_i}$ on $\mathcal{O}_M(P_i)$, $i = 1, \dots, N$ by modifying $\rho_{\text{Ar}, \omega_{\text{hyp}}, P_i}$'s. In other words, from now on, we assume that the (resulting constant) ratio

$$C_{\text{hyp}}^i := e^{c_{\text{hyp}}^i} := \rho_{\text{hyp};P_i} / \rho_{\text{Ar}, \omega_{\text{hyp}}, P_i} \quad (6.2.3)$$

does not depend on i . Obviously, with all this, the condition in (6.1.2), claiming that $\underline{K_M(P_1 + \dots + P_N)}_{\omega_{\text{hyp}}}$ multiplying by $e^{a_{\text{hyp}} + c_{\text{hyp}}^1 + \dots + c_{\text{hyp}}^N}$ is isometric to $K(P_1 + \dots + P_N)$ together with the natural metric $\tau_{\text{hyp}; K_M(P_1 + \dots + P_N)}^\vee$ induced from τ_{hyp} on M^0 , determines the constant $c_{\text{hyp}} := c_{\text{hyp}}^i$, $i = 1, \dots, N$ and hence the metrics $\rho_{\text{hyp};P_i}$ on $\mathcal{O}_M(P_i)$, $i = 1, \dots, N$, uniquely. This then finishes our discussion on how to impose geometrically meaningful admissible metrics on K_M and on the cuspidal line bundles respectively. For our own convinences, we set

$$\underline{\mathcal{O}_M(P_i)}_{\text{hyp}} := (\mathcal{O}_M(P_i), \rho_{\text{hyp};P_i}), \quad i = 1, \dots, N. \quad (6.2.4)$$

Remark 6.2.1. Here we omit the condition (6.1.3), as this can hardly be the case. Nevertheless, by the above discussion, we see that the ratio of the metrics on both hands of (6.1.3) is a constant which depends only on punctured Riemann surface M^0 itself.

(6.3) Before finally giving the geometric interpretation for our metric on determinant of cohomology, we in this subsection using the result in (5.4) give the Mumford type isometry associated to hyperbolic metrics, which will be used in the next section on Takhtajan-Zograf metrics.

For this purpose, we apply Theorem 5.4.1 as follows. First of all, take ω to be the normalized hyperbolic volume ω_{hyp} . Secondly, set $(\alpha; \beta_1, \dots, \beta_N)$ in subsection (5.4) to be $(a_{\text{Ar}, \text{hyp}}; c_{\text{hyp}}^1, \dots, c_{\text{hyp}}^N)$ introduced in (6.2). Finally, denote the resulting corresponding Hermitian line bundles by the underline with the lower index hyp, e.g., $\underline{\lambda}_{n_{\text{hyp}}}$ stands for $\overline{\lambda}_n^{\alpha; \beta}$, $\underline{\Delta}_{n_{\text{hyp}}}$ stands for $\overline{\Delta}_n^{\alpha; \beta}$, etc.. Then we have the following

Theorem 6.3.1. (Fundamental Theorem with respect to Hyperbolic Metrics)
With the same notation as above, for any positive integer n , we have the following isometries:

(a) (Serre isometry)

$$\underline{\lambda}_{n_{\text{hyp}}} \simeq \underline{\lambda}_{1-n_{\text{hyp}}};$$

(b) (*Generalized Mumford isometry*)

$$\underline{\lambda}_{n\text{hyp}}^{\otimes 12} \simeq \underline{\Delta}_{0\text{hyp}}^{\otimes 6n^2 - 6n + 1} \otimes \underline{\Delta}_{1\text{hyp}} \otimes \underline{\Delta}_{2\text{hyp}}^{\otimes -12n + 10} \otimes \mathcal{O}(e^{a(q)});$$

(c) (*Generalized Mumford isometry*)

$$\underline{\lambda}_{n\text{hyp}} \simeq \underline{\lambda}_{1\text{hyp}}^{\otimes 6n^2 - 6n + 1} \otimes \underline{\Delta}_{1\text{hyp}}^{\otimes -\frac{n(n-1)}{2}} \otimes \underline{\Delta}_{2\text{hyp}}^{\otimes (n-1)^2} \otimes \mathcal{O}(e^{-\frac{n(n-1)}{2} \cdot a(q)}).$$

Obviously, even though we only discuss our metric theory for a single curve, but the technique can be globalized so that we can apply the above discussion for a family of curves. In particular, this then works over the Teichmüller space $T_{q,N}$ of punctured Riemann surfaces of signature (q, N) as well as over the moduli space $\mathcal{M}_{q,N}$ of punctured Riemann surfaces of signature (q, N) . Moreover, as

$$\underline{K}_M(P_1 + \cdots + P_N)_{\text{hyp}} \simeq (K_M(D), \tau_{\text{hyp}; K_M(D)}^\vee), \quad (6.3.1)$$

by a work of Wolpert ([Wo1]), (see e.g. [TZ2] and (7.2) below for the detail,) we may deduce that

$$c_1(\underline{\Delta}_{0\text{hyp}}) = \frac{\omega_{\text{WP}}}{\pi^2}. \quad (6.3.2)$$

Here ω_{WP} denotes the Weil-Petersson Kähler form. Thus in particular, we have the following

Corollary 6.3.2. *With the same notation as above, for all positive integers n , we have the following identities of $(1,1)$ -forms on $T_{q,N}$ and hence on $\mathcal{M}_{q,N}$:*

$$12 c_1(\underline{\lambda}_{n\text{hyp}}) = (6n^2 - 6n + 1) \frac{\omega_{\text{WP}}}{\pi^2} + c_1(\underline{\Delta}_{1\text{hyp}}) - (12n - 10) c_1(\underline{\Delta}_{2\text{hyp}}). \quad (6.3.3)$$

Theorem 6.3.1 and Corollary 6.3.2 will be used to connect our work with the beautiful pioneer work of Takajan and Zograf ([TZ1,2]) in §7.

(6.4) The geometric interpretation of our metrics on determinant of cohomology at this stage is given in terms of the new metric on $\lambda(K_M)$ with respect to the hyperbolic metric.

Realize M^0 as a quotient $\Gamma \backslash \mathcal{H}$ of the upper half-plane by the action of a torsion free finitely generated Fuchsian group Γ . Then it is well-known that we may choose $\Gamma \subset PSL(2, \mathbb{R})$ to be a subgroup generated by $2q$ hyperbolic transformations

$A_1, B_1, \dots, A_q, B_q$ and N parabolic transformations S_1, \dots, S_N satisfying the single relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \dots A_q B_q A_q^{-1} B_q^{-1} S_1 \dots S_N = 1.$$

Choose a normalized basis of abelian differentials ψ_1, \dots, ψ_q , i.e., a basis of the vector space $H^0(M, K_M)$ so that

$$\int_z^{A_i z} \psi_j(w) dw = \delta_{ij}, \quad \int_z^{B_i z} \psi_j(w) dw =: \tau_{ij}, \quad i, j = 1, \dots, q,$$

with δ_{ij} the Kronecker symbol and $\tau = (\tau_{ij})$ the period matrix of M .

On $\lambda(K_M)$, choose the section $(\psi_1 \wedge \dots \wedge \psi_q) \otimes 1^\vee$, with 1 the canonical section of $H^1(M, K_M) \simeq \mathbb{C}$. Then we have the following

Theorem 6.4.1. *With the same notation as above, as the metric on $\lambda(K_M)$,*

$$\begin{aligned} & \langle (\psi_1 \wedge \dots \wedge \psi_q) \otimes 1^\vee, (\psi_1 \wedge \dots \wedge \psi_q) \otimes 1^\vee \rangle_{h_{\underline{K}_{M_{\text{hyp}}}}(\underline{K}_{M_{\text{hyp}}})} \\ &= \left(\det(\text{Im} \tau) \cdot 2\pi(2q-2) \right) \cdot (\det^*(\Delta_{\text{hyp}}))^{-1}. \end{aligned}$$

Proof. From §5, we know that the new metrics on determinant of cohomology obey the rules in §3 for the Riemann-Roch metrics. Thus we see that

$$\begin{aligned} & h_{\underline{K}_{M_{\text{hyp}}}}(\underline{K}_{M_{\text{hyp}}}) \\ &= h_{\underline{K}_{M_{\omega_{\text{hyp}}}}}(\underline{K}_{M_{\omega_{\text{hyp}}}}) \cdot e^{\frac{2q-2}{12} \cdot a_{\text{Ar}, \text{hyp}}} \quad (\text{by (3.2.4)}) \\ &= h_{\underline{K}_{M_{\omega_{\text{hyp}}}}}(\underline{K}_{M_{\omega_{\text{hyp}}}}) \cdot e^{\frac{2q-2}{12} \cdot \frac{12}{2} \cdot \frac{1}{2q-2} \cdot \left(\log \frac{\det^* \Delta_{\text{Ar}}}{A_{\text{Ar}}(M)} - \log \frac{\det^* \Delta_{\text{hyp}}}{2\pi(2q-2)} \right)} \\ & \quad (\text{by (6.1.8)}) \\ &= h_{\underline{K}_{M_{\omega_{\text{hyp}}}}}(\underline{K}_{M_{\omega_{\text{hyp}}}}) \cdot \sqrt{\frac{\det^* \Delta_{\text{Ar}}}{A_{\text{Ar}}(M)}} \cdot \left(\sqrt{\frac{\det^* \Delta_{\text{hyp}}}{2\pi(2q-2)}} \right)^{-1} \\ &= \left(h_{\underline{K}_{M_{\omega_{\text{Ar}}}}}(\underline{K}_{M_{\omega_{\text{Ar}}}}) \cdot \sqrt{\frac{\det^* \Delta_{\text{Ar}}}{A_{\text{Ar}}(M)}} \right) \cdot \left(\sqrt{\frac{\det^* \Delta_{\text{hyp}}}{2\pi(2q-2)}} \right)^{-1} \\ & \quad (\text{by (3.3.2)}). \end{aligned}$$

But we know that $h_{\underline{K}_{M_{\omega_{\text{Ar}}}}}(\underline{K}_{M_{\omega_{\text{Ar}}}})$ is simply the Quillen metric on $\lambda(K_M)$ with respect to the Arakelov metric, thus, by definition,

$$h_{\underline{K}_{M_{\omega_{\text{Ar}}}}}(\underline{K}_{M_{\omega_{\text{Ar}}}}) = h_{F; \omega_{\text{Ar}}} \cdot \left(\sqrt{\frac{\det^* \Delta_{\text{Ar}}}{A_{\text{Ar}}(M)}} \right)^{-1}.$$

This to say,

$$h_{\underline{K}_{M_{\omega_{\text{Ar}}}}}(\underline{K}_{M_{\omega_{\text{Ar}}}}) \cdot \sqrt{\frac{\det^* \Delta_{\text{Ar}}}{A_{\text{Ar}}(M)}}$$

is simply the Faltings metric $h_{F;\omega_{Ar}}$ on $\lambda(K_M)$, which is nothing but the determinant of the L^2 -pairing on $H^0(M, K_M)$. Therefore, by Serre duality, we see that

$$\begin{aligned} & \langle (\psi_1 \wedge \cdots \wedge \psi_q) \otimes 1^\vee, (\psi_1 \wedge \cdots \wedge \psi_q) \otimes 1^\vee \rangle_{\underline{K}_{M_{\text{hyp}}}(\underline{K}_{M_{\text{hyp}}})} \\ &= \langle (\psi_1 \wedge \cdots \wedge \psi_q) \otimes 1^\vee, (\psi_1 \wedge \cdots \wedge \psi_q) \otimes 1^\vee \rangle_{h_{F;\omega_{Ar}}} \cdot \left(\sqrt{\frac{\det^* \Delta_{\text{hyp}}}{2\pi(2q-2)}} \right)^{-2} \\ &= \left(\det \text{Im} \tau \cdot 2\pi(2q-2) \right) \cdot (\det^*(\Delta_{\text{hyp}}))^{-1}. \end{aligned}$$

This completes the proof of the theorem.

As a direct consequence of the proof of Theorem 6.4.1, we have the following

Corollary 6.4.2. *With the same notation as above, if $M^0 = M$, i.e., if M^0 is compact, then $\underline{K}_{M_{\text{hyp}}}$ is nothing but K_M together with standard (smooth) hyperbolic metric. In other words, when the Riemann surface is compact, then the Arakelov-Poincaré volume is simply the ratio between the standard (smooth) hyperbolic metric and the Arakelov metric with respect to the normalized hyperbolic volume form.*

Remark 6.4.1. Recall that by various Mean Value Lemmas, for our metric theory, the ω -Arakelov metric is essentially the original Arakelov metric, which is in the nature of Euclidean geometry. Hence, the above corollary shows that the Arakelov-Poincaré volume indeed measures how far the Euclidean aspect of a compact Riemann surface is away from its Poincaré aspect.

§7. On Takhtajan-Zograf metric over moduli space of punctured Riemann surfaces

(7.1) To facilitate ensuing discussion on an application of our metric, we in this subsection recall some results of Takhtajan and Zograf ([TZ1,2]).

For a punctured Riemann surface M^0 of signature (q, N) (with $2q + N \geq 3$), let Γ be a torsion free Fuchsian group uniformizing M^0 , i.e., $M^0 \simeq \Gamma \backslash \mathcal{H}$, where \mathcal{H} denotes the complex upper-half plane. Denote by $\Gamma_1, \dots, \Gamma_N$ the set of non-conjugate parabolic subgroups in Γ , and for every $i = 1, \dots, N$, fix an element $\sigma_i \in PSL(2, \mathbb{R})$ such that $\sigma_i^{-1} \Gamma_i \sigma_i = \Gamma_\infty$, where the group Γ_∞ is generated by the parabolic transformation $z \mapsto z + 1$. As usual, define the Eisenstein series $E_i(s, z)$ corresponding to the i -th cusp of the group Γ for $\text{Re}(s) > 1$ by

$$E_i(s, z) := \sum_{\gamma \in \Gamma_i \backslash \Gamma} \text{Im}(\sigma_i^{-1} \gamma z)^s, \quad i = 1, \dots, N. \quad (7.1.1)$$

Denote the Teichmüller space of punctured Riemann surfaces of signature (q, N) by $T_{q,N}$. Then at the point $[M^0]$ corresponding to a punctured Riemann surface M^0 , the tangent space $T_{[M^0]}T_{q,N}$ can be naturally identified with the space $\Omega^{-1,1}(M^0)$ of harmonic L^2 -tensors on M^0 of type $(-1,1)$. Define the Weil-Petersson metric on $T_{q,N}$ by

$$\langle \psi, \psi \rangle_{\text{WP}} := \int_{M^0} \psi \bar{\psi} d\mu_{\text{hyp}}, \quad (7.1.2)$$

where $\psi, \psi \in \Omega^{-1,1}(M^0)$ are considered as tangent vectors of $T_{q,N}$ at $[M^0]$, and $d\mu_{\text{hyp}} = 2\pi(2q - 2 + N) \omega_{\text{hyp}}$ is the Kähler form corresponding to the uniformizing hyperbolic metric τ_{hyp} induced from $\mathcal{H} \rightarrow \Gamma \backslash \mathcal{H} \simeq M^0$ with Gaussian curvature -1.

Following Takhtajan and Zograf, for $i = 1, \dots, N$, define the metric \langle, \rangle_i on $T_{q,N}$ by setting

$$\langle \phi, \psi \rangle_i := \int_{M^0} \phi \bar{\psi} E_i(\cdot, 2) d\mu_{\text{hyp}}, \quad \phi, \psi \in \Omega^{-1,1}(M^0). \quad (7.1.3)$$

Here $E_i(z, s)$ is the Eisenstein series defined in (7.1.1). In [TZ2], it is proved that \langle, \rangle_i , $i = 1, \dots, N$, are Kähler metrics on $T_{q,N}$. Moreover, their sum $\sum_{i=1}^N \langle, \rangle_i$ gives a new Kähler metric $\langle, \rangle_{\text{cusp}}$, the cusp metric or the Takhtajan-Zograf metric, on $T_{q,N}$, which is invariant under the action of the Teichmüller modular group. Denote the corresponding Kähler form by ω_{cusp} , or ω_{TZ} .

For compact Riemann surfaces M , a work of D'Hoker-Phong [D'HP] and Sarnak [Sa] shows that the regularized determinant $\det^* \Delta_n$ associated to $K_M^{\otimes n}$ with respect to hyperbolic metrics defined via the zeta function formalism of Ray-Singer, is equal, up to a constant multiplier depending only on q and n , to $Z'_M(1)$ for $n = 0, 1$, and $Z_M(n)$ for $n \geq 2$. Here $Z_M(s)$ denotes the Selberg zeta function associated to M . Motivated by this and the Quillen metric on determinant of cohomology, for punctured Riemann surfaces, Takhtajan and Zograf ([TZ1,2]) define $\det_{\text{TZ}}^* \Delta_n$ with respect to hyperbolic metrics by simply setting

$$\det_{\text{TZ}}^* \Delta_n := \begin{cases} Z'_{M^0}(1), & \text{if } n = 0, 1; \\ Z_{M^0}(n), & \text{if } n \geq 2. \end{cases}$$

Here $Z_{M^0}(s)$ denotes the Selberg zeta function of M^0 defined in (6.1.4). Moreover, for any $n \in \mathbb{Z}_{\geq 1}$, on $\lambda_n := \lambda(K_M^{\otimes n} \otimes \mathcal{O}_M(P_1 + \dots + P_N)^{\otimes(n-1)})$, they introduce the norm $h_{\text{TZ},n}$ by setting

$$h_{\text{TZ},n} := h_P \cdot \det_{\text{TZ}}^* \Delta_n^{-\frac{1}{2}},$$

where h_P denotes the determinant of Petersson norm on λ_n .

Theorem 7.1.1. ([TZ1,2]) *With the same notation as above, as $(1,1)$ forms on $T_{q,N}$:*

$$c_1(\lambda_n, h_{\text{TZ},n}) = \frac{6n^2 - 6n + 1}{12} \frac{\omega_{\text{WP}}}{\pi^2} - \frac{1}{9} \omega_{\text{TZ}}. \quad (7.1.4)$$

Note that $Z_{M^0}(s)$ has a simple zero at $s = 1$, our definition (6.1.7) of the regularized determinant of Laplacian Δ_{hyp} differs from Takhtajan and Zograf's $\det_{\text{TZ}}^* \Delta_1$ up to a universal constant factor depending only on the signature (q, N) . (See e.g., Remark 6.1.2.) Therefore, by Theorem 6.4.1, we have the following

Theorem 7.1.1'. *With the same notation as above, as $(1,1)$ forms on $T_{q,N}$:*

$$12 \cdot c_1(\lambda_{1_{\text{hyp}}}) = \frac{\omega_{\text{WP}}}{\pi^2} - \frac{4}{3} \omega_{\text{TZ}}. \quad (7.1.5)$$

Moreover, as all bundles, forms, and metrics are invariant under the action of the Teichmüller modular group, (7.1.4) and (7.1.5) actually induce the same relations on the moduli space $\mathcal{M}_{q,N}$ of punctured Riemann surfaces of signature (q, N) (in the sense of V -manifolds).

(7.2) Now let us look at Theorem 6.3.1(b) and (c) carefully. First of all, by definition in (6.1), we know that $\overline{K_M}^{a_{\text{hyp}}}(\overline{P_1}^{c_{\text{hyp}}} + \cdots + \overline{P_N}^{c_{\text{hyp}}})$ is really nothing but $K_M(P_1 + \cdots + P_N)$ together with the hyperbolic metric $\tau_{\text{hyp}; K_M(P_1 + \cdots + P_N)}^\vee$ naturally induced from the hyperbolic volume form $d\mu_{\text{hyp}}$. Thus in particular, over $T_{q,N}$, the Teichmüller space of punctured Riemann surfaces of signature (q, N) , the first Chern form for the metrized Deligne pairing

$$\langle \overline{K_M}^{a_{\text{hyp}}}(\overline{P_1}^{c_{\text{hyp}}} + \cdots + \overline{P_N}^{c_{\text{hyp}}}), \overline{K_M}^{a_{\text{hyp}}}(\overline{P_1}^{c_{\text{hyp}}} + \cdots + \overline{P_N}^{c_{\text{hyp}}}) \rangle$$

may be naturally associated to $\frac{\omega_{\text{WP}}}{\pi^2}$ by the work of Wolpert as stated in [TZ2]. In fact, we have the following

Theorem 7.2.1. ([Wo], [TZ2]) *With the same notation as above,*

$$\int c_1(K_M(P_1 + \cdots + P_N), \tau_{\text{hyp}; K_M(P_1 + \cdots + P_N)}^\vee)^2 = \frac{\omega_{\text{WP}}}{\pi^2}.$$

On the other hand, by arithmetic intersection theory, we have

$$c_1(\langle \bar{L}, \bar{L}' \rangle) = \int c_1(\bar{L}) \cdot c_1(\bar{L}').$$

(See e.g., [El], where the metrics involved are supposed to be smooth. But this restriction can be easily removed to apply here, as the singularities of our metrics on admissible metrized line bundles are of hyperbolic growth.) Hence,

$$c_1\left(\langle \overline{K_M}^{a_{\text{hyp}}}(\overline{P_1}^{c_{\text{hyp}}} + \cdots + \overline{P_N}^{c_{\text{hyp}}}), \overline{K_M}^{a_{\text{hyp}}}(\overline{P_1}^{c_{\text{hyp}}} + \cdots + \overline{P_N}^{c_{\text{hyp}}}) \rangle\right) = \frac{\omega_{\text{WP}}}{\pi^2}.$$

As a direct consequence, as claimed in Corollary 6.3.2,

$$12c_1(\lambda_{n_{\text{hyp}}}) = (6n^2 - 6n + 1)\frac{\omega_{\text{WP}}}{\pi^2} + c_1(\underline{\Delta}_{1_{\text{hyp}}}) - (12n - 10)c_1(\underline{\Delta}_{2_{\text{hyp}}}). \quad (7.2.1)$$

Moreover, by the generalized Mumford isometry,

$$\left(\lambda_{n_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes n-1}\right)^{\otimes 12} \simeq \underline{\Delta}_{0_{\text{hyp}}}^{\otimes 6n^2-6n+1} \otimes \underline{\Delta}_{1_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes -2} \otimes \mathcal{O}(e^{a(q)}) \quad \text{for all } n \geq 1. \quad (7.2.2)$$

Here

$$\underline{\Delta}_{0_{\text{hyp}}} := \langle \underline{K_N}(\mathbb{P}_1 + \cdots + \mathbb{P}_N)_{\text{hyp}}, \underline{K_N}(\mathbb{P}_1 + \cdots + \mathbb{P}_N)_{\text{hyp}} \rangle(\pi_N).$$

(7.3) Put (7.2.2) in the language of differential forms, we see that, if $n \geq 1$,

$$12c_1\left(\lambda_{n_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes n-1}\right) = (6n^2 - 6n + 1)c_1(\underline{\Delta}_{0_{\text{hyp}}}) + c_1(\underline{\Delta}_{1_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes -2}). \quad (7.3.1)$$

Let $n = 1$, and use Theorem 7.1.2 and Theorem 7.2.1, we have the following

Theorem 7.3.1. (Fujiki-Weng) *With the same notation as above,*

$$\frac{4}{3}\omega_{\text{TZ}} = c_1\left(\underline{\Delta}_{1_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes -2}\right).$$

So $\frac{4}{3}\omega_{\text{TZ}}$ can be realized as the first Chern form of the metrized line bundle $\underline{\Delta}_{1_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes -2}$. In particular, the Takhtajan-Zograf metric is Kähler and $\frac{4}{3}\omega_{\text{TZ}}$ is a Hodge metric form.

Remark 7.3.1. Note that as a line bundle Δ_2 is indeed trivial. So if we only interested in the results in the bundle version, we in fact have the following simple relation

$$\lambda_n^{\otimes 12} \simeq \Delta_0^{\otimes 6n^2-6n+1} \otimes \Delta_1. \quad (7.3.2)$$

On the other hand, arithmetically, $\underline{\Delta}_{2_{\text{hyp}}}$ is far from being trivial. It seems to be equally interesting to study the associated smooth function on $\mathcal{M}_{q,N}$ resulting from the corresponding metric on $\underline{\Delta}_{2_{\text{hyp}}}$.

Theorem 7.3.2. (Fundamental Relations for Riemann Surfaces) *With the same notation as above, $\mathcal{M}_{q,N}$, for $n \geq 0$, there exists the canonical isometry, up to a constant factor depending only q, N and n ,*

$$(\lambda_n, h_{\text{TZ},n})^{\otimes 12} \simeq \underline{\Delta}_{0_{\text{hyp}}}^{\otimes 6n^2-6n+1} \otimes \left(\underline{\Delta}_{1_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes -2}\right). \quad (7.3.3)$$

In particular, up to a constant factor depending only q, N and n ,

$$(\lambda_n, h_{\text{TZ},n}) = \underline{\lambda}_{n_{\text{hyp}}} \otimes \underline{\Delta}_{2_{\text{hyp}}}^{\otimes n-1}. \quad (7.3.4)$$

Proof. Easily we see that by (7.3.1) and Theorem 7.3.1, both sides of (7.3.3) have the same first Chern form over $\mathcal{M}_{q,N}$. Thus by the structure of the stably compactified moduli space $\overline{\mathcal{M}}_{q,N}$, the metrics on both sides are proportional to each other. (See e.g. the proof of Theorem A2.4.1 in the appendix.) This completes the proof of the theorem.

APPENDIX:

Arithmetic Factorization Theorem in terms of Intersection

In this appendix, we propose an arithmetic factorization for Weil-Petersson geometry, Takhtajan-Zograf geometry and Selberg geometry associated to punctured Riemann surfaces. Unlike the rest of this paper, the discussion here is rather informal, in particular, not so many rigorous proofs are given for the assertions. So for the time being, the reader may simply understand them as some working hypothesis. On the other hand, we anticipate that this arithmetic factorization will play a key role in studying the global geometry of moduli spaces of Riemann surfaces.

§A1. Degeneration of Weil-Petersson metrics

(A1.1) We start with Masur's result on degeneration of Weil-Petersson metrics. Let \mathcal{M}_q be the moduli space of compact Riemann surfaces of genus $q \geq 2$. Denote its stably compactification by $\overline{\mathcal{M}}_q$. Let $p \in E := \overline{\mathcal{M}}_q \setminus \mathcal{M}_q$ be a boundary point and let U be a small neighborhood of p . Let $\pi : \Delta^n \rightarrow \Delta^n/G = U$ be a local uniformizing chart with holomorphic coordinates $((z_i)_{i=1}^r, (w_j)_{j=1}^{3q-3-r})$ such that $\pi((0), (0)) = p$, and $\pi^{-1}(U \cap E) = \cup_i z_i^{-1}(0)$. As before, denote by ω_{WP} the Weil-Petersson Kähler form. Write

$$\begin{aligned} & \pi^*(\omega_{\text{WP}}|_{U \cap \mathcal{M}_q}) \\ &= \sqrt{-1} \sum a_{i\bar{j}} dz_i \wedge d\bar{z}_j - 2\text{Im} \sum b_{i\bar{k}} dz_i \wedge d\bar{w}_k + \sqrt{-1} \sum c_{k\bar{l}} dw_k \wedge d\bar{w}_l. \end{aligned}$$

Then we have the following fundamental result of Masur ([Ma]):

Theorem A1.1.1. (Masur) *For $((z_i), (w_k))$ near 0,*

$$(i) \quad \frac{C^{-1}}{|z_i|^2(-\log|z_i|)^3} \leq a_{i\bar{i}} \leq \frac{C^{-1}}{|z_i|^2(-\log|z_i|)^3}, \text{ for a constant } C > 0;$$

$$(ii) \ a_{i\bar{j}} = O\left(\frac{1}{|z_i||z_j|(-\log|z_i|)^3(-\log|z_j|)^3}\right), \text{ if } i \neq j;$$

$$(iii) \ b_{i\bar{k}} = O\left(\frac{1}{|z_i|(-\log|z_i|)^3}\right);$$

$$(iv) \ \lim_{((z_i), (w_k)) \rightarrow 0} c_{k\bar{l}} = h_{k\bar{l}} \text{ for a certain constant positive definite hermitian matrix } (h_{k\bar{l}}).$$

(A1.2) Previously, when mathematicians talked about Masur's result, they usually paid much more attention on the first three conclusions, i.e., the asymptotic behavior. Even though such an asymptotic behavior is very important geometrically, but it has little arithmetic meaning. On the other hand, there is another very important part which is hidden in conclusion (iv) – this last statement clearly indicate that the restriction of the Weil-Petersson Kähler form to the boundary could result a metric on the boundary. Indeed, later we will see that it is not unreasonable to expect that this induced form should coincide with the Weil-Petersson Kähler form for the boundary.

(A1.3) To justify what we said in (A1.2), we next recall yet another fundamental result due to Wolpert ([Wo2]).

Let $\overline{\pi}_q : \overline{\mathcal{C}}_q \rightarrow \overline{\mathcal{M}}_q$ (resp. $\pi_q : \mathcal{C}_q \rightarrow \mathcal{M}_q$) be the universal curve over the stably moduli space (resp. moduli space) of compact Riemann surfaces. Then it is well-known that the standard hyperbolic metric on compact Riemann surfaces may be glued together to give a smooth metric on the relative canonical line bundle K_{π_q} . On the other hand, even on singular fibers of $\overline{\pi}_q$, we may get standard hyperbolic metrics on the corresponding punctured Riemann surfaces. A natural question is whether these (singular) hyperbolic metrics can also be glued together so that we can get a certain type of metric on the relative canonical line bundle $K_{\overline{\pi}_q}$. The answer is yes. In fact, we have the following

Theorem A1.3.1. (Wolpert) *With the same notation as above, the resulting metric on the relative canonical line bundle $K_{\overline{\pi}_q}$ obtaining from standard hyperbolic metrics on the fibers is continuous and good.*

As a direct consequence of this result, Wolpert then deduces that, in the sense of currents, on the compactified moduli space $\overline{\mathcal{M}}_q$, $\frac{\omega_{WP}}{\pi^2}$ is the curvature form of a continuous metric h_{WP} on a certain line bundle and the metric h_{WP} may be approximated by smooth positive curvature metrics. We later will give an alternative proof of this statement.

(A1.4) Motivated by Wolpert's result, in [TW2], we study how to glue admissible metrics along with a degeneration family of compact Riemann surfaces. This roughly goes as follows.

To facilitate ensuing discussion, we first recall the plumbing construction of a degenerating family of Riemann surfaces starting from M as follows (cf. e.g. [Fay1], [Ma] and [Wo2]). Let $M^0 := M \setminus \{p\}$. Then M^0 is a punctured Riemann surface with two punctures p_1, p_2 in place of p , where p_1, p_2 correspond to two points in the normalization \tilde{M} of M . Denote the unit disc in \mathbb{C} by Δ . For $i = 1, 2$, fix a coordinate function $z_i : U_i \rightarrow \Delta$ such that $z_i(p_i) = 0$, where U_i is an open neighborhood of p_i . For each $t \in \Delta$, let $S_t := \{(x, y) \in \Delta^2 : xy = t\}$. Now for each $t \in \Delta$, remove the discs $|z_i| < |t|$, $i = 1, 2$, from M and glue the remaining surface with S_t via the identification

$$z_1 \sim (z_1, t/z_1) \quad \text{and} \quad z_2 \sim (t/z_2, z_2). \quad (\text{A1.4.1})$$

The resulting surfaces $\{M_t\}_{t \in \Delta}$ form an analytic family $\pi : \mathcal{M} \rightarrow \Delta$ with $M_0 = M$. Here π denotes the holomorphic projection map. Note that for $t \neq 0$, each fiber M_t is a compact Riemann surface of genus g . Also the node p does not disconnect the Riemann surface when removed from M . The restriction of $\ker(d\pi)$ to $\mathcal{M} \setminus \{p\}$ forms a holomorphic line bundle over $\mathcal{M} \setminus \{p\}$ such that $L|_{M_t} = TM_t$ and $L|_{M^0} = TM^0$, which will be called the vertical line bundle. Note that $\ker(d\pi)$ itself does not form a line bundle over \mathcal{M} since $\ker(d\pi)$ is of rank 2 at p . Similarly, one may construct a degenerating family of compact Riemann surfaces such that the center fiber is a nodal curve with a separating node.

Now we are ready to state the following result of To and myself in [TW2].

Let $\{M_t\}$ be a family of compact Riemann surface of genus $g \geq 2$ degenerating to a Riemann surface M of genus $g - 1$ with a single node p as described above. Let $L = \{L_t\}$ be a line bundle on $\{M_t\}$. Then

(i) in the case when M_0 is with a non-separating node, there is a continuous metric ρ defined everywhere on $\{M_t\}$, except possibly at the node, such that

(a) the restriction of ρ to $\{M_t\}_{t \neq 0}$ is smooth;

(b) for each $t \neq 0$, the restriction of ρ to L_t is $d\mu_{\text{hyp}, t}$ -admissible;

(ii) in the case when M_0 is with a separating node, the following two conditions are equivalent:

(A) there is a continuous metric ρ defined everywhere on $\{M_t\}$, except possibly at the node, such that

(a) the restriction of ρ to $\{M_t\}_{t \neq 0}$ is smooth;

(b) for each $t \neq 0$, the restriction of ρ to L_t is $d\mu_{\text{hyp},t}$ -admissible;

(B) the degrees d_1 and d_2 of L_0 on the irreducible components $M_0^{(1)}$ of genus q_1 and $M_0^{(2)}$ of genus q_2 of M_0 satisfy

$$d_1(2q_2 - 1) = d_2(2q_1 - 1).$$

Moreover we know that, in general, if we do not have the degree condition as in (B), admissible metrics on $\{L_t\}_{t \neq 0}$ will continuously extend to one of the irreducible component, while blow up to infinity on the other irreducible component.

§A2. Arithmetic Factorization Theorem: a proposal

(A2.1) We start with an algebraic factorization theorem. So we go back to study the universal curve $\overline{\pi_{q,N}} : \overline{\mathcal{C}_{q,N}} \rightarrow \overline{\mathcal{M}_{q,N}}$. On $\overline{\mathcal{C}_{q,N}}$, the following line bundles are well-defined: K , the relative canonical line bundle; $\mathcal{O}(\mathbb{P}_i)$, $i = 1, \dots, N$, N sections. Hence, by using Deligne pairing formalism, we get the following line bundles over $\overline{\mathcal{M}_{q,N}}$: $\langle K(\mathbb{P}_1 + \dots + \mathbb{P}_N), K(\mathbb{P}_1 + \dots + \mathbb{P}_N) \rangle$; $\langle K(\mathbb{P}_1 + \dots + \mathbb{P}_N), \mathbb{P}_i \rangle$; $\langle K, \mathbb{P}_i \rangle$, $i = 1, \dots, N$. Parallely, we have Mumford type line bundles λ_n introduced in §5. Moreover, we know that these line bundles on $\overline{\mathcal{M}_{q,N}}$ satisfy Mumford type relations, i.e., Theorem 5.2.2 (on $\mathcal{M}_{q,N}$).

Now we want to know how these bundles change when we restrict them to the boundary of $\overline{\mathcal{M}_{q,N}}$, or better when we pull back these bundles via the normalization of the stable curves. For simplicity, we only study the case when one more non-separating node is involved. So we have the following natural map $\alpha : \mathcal{M}_{q-1,N+2} \rightarrow \overline{\mathcal{M}_{q,N}}$. We will use \tilde{K} to denote the relative canonical line bundle for the universal curve on $\mathcal{M}_{q-1,N+2}$, and use $\tilde{\mathbb{P}}_i$, $i = 1, \dots, N$ and \mathbb{R}, \mathbb{S} to denote $N + 2$ sections (so that \mathbb{P}_i corresponds to $\tilde{\mathbb{P}}_i$, $i = 1, \dots, N$ and \mathbb{R}, \mathbb{S} are two more sections corresponding to the non-separating node for the restriction of the original universal curve $\overline{\pi_{q,N}}$ to the boundary.)

Obviously, in this case, we have the following algebraic factorization:

(a) the line bundle $\langle K(\mathbb{P}_1 + \dots + \mathbb{P}_N), K(\mathbb{P}_1 + \dots + \mathbb{P}_N) \rangle$ changes to $\langle \tilde{K}(\tilde{\mathbb{P}}_1 + \dots + \tilde{\mathbb{P}}_N + \mathbb{R} + \mathbb{S}), \tilde{K}(\tilde{\mathbb{P}}_1 + \dots + \tilde{\mathbb{P}}_N + \mathbb{R} + \mathbb{S}) \rangle$;

(b) the line bundle $\langle K(\mathbb{P}_1 + \cdots + \mathbb{P}_N), \mathbb{P}_i \rangle$ changes to $\langle \tilde{K}(\tilde{\mathbb{P}}_1 + \cdots + \tilde{\mathbb{P}}_N + \mathbb{R} + \mathbb{S}), \tilde{\mathbb{P}}_i \rangle$, $i = 1, \dots, N$;

(c) the line bundle $\langle K, \mathbb{P}_i \rangle$ changes to $\langle \tilde{K}, \tilde{\mathbb{P}}_i \rangle \otimes \langle \mathbb{R} + \mathbb{S}, \tilde{\mathbb{P}}_i \rangle$, $i = 1, \dots, N$;

(d) the line bundle $\lambda_n^{\otimes 12} \otimes (\langle K(\mathbb{P}_1 + \cdots + \mathbb{P}_N), \mathbb{P}_1 + \cdots + \mathbb{P}_N \rangle)^{\otimes 6(n-1)}$ changes to $\tilde{\lambda}_n^{\otimes 12} \otimes \left(\langle \tilde{K}(\tilde{\mathbb{P}}_1 + \cdots + \tilde{\mathbb{P}}_N + \mathbb{R} + \mathbb{S}), \tilde{\mathbb{P}}_1 + \cdots + \tilde{\mathbb{P}}_N + \mathbb{R} + \mathbb{S} \rangle \right)^{\otimes 6(n-1)} \otimes \left(\langle \tilde{K}, \mathbb{R} + \mathbb{S} \rangle \otimes \langle \tilde{K}(\mathbb{R} + \mathbb{S}), \mathbb{R} + \mathbb{S} \rangle \right)$.

The proof may be obtained by looking at the intersection first, which gives (a), (b) and (c). With (a), (b) and (c), (d) is a direct consequence of the generalized Mumford relation from Theorem 5.2.2. In fact note that many components in (d) are trivial line bundles, we may rewrite (d) as

(d') the line bundle $\lambda_n^{\otimes 12}$ changes to $\tilde{\lambda}_n^{\otimes 12} \otimes \langle \tilde{K}, \mathbb{R} + \mathbb{S} \rangle$.

(A2.2) With the above discussion, we may now offer the following global picture for the geometry of punctured Riemann surfaces, or better, for the geometry of moduli spaces of punctured Riemann surfaces.

First of all, our generalized Mumford type isometrie in Theorem 6.3.1, together with Theorem 7.3.1 and Theorem 7.3.2 expose explicitly the intrinsic relations among the spectrum geometry given by Selberg zeta functions, the deformation geometry given by Weil-Petersson metric, and the cusp geometry given by Eisenstein series via Takhtajan-Zograf metrics.

Secondly, the deformation geometry and the cusp geometry are in the nature of arithmetic intersection theory. So their properties should be relatively easier to understand. As a consequence, via Mumford isometries, we then could get information about the spectrum geometry, which is in nature of cohomology theory.

Finally, algebraic factorization (a) shows that via the degeneration and normalization process, the Weil-Petersson geometry factors extremely well. So the arithmetic counter part should be established in a rather formal way. Similarly, we can apply this comment to the cusp geometry by looking at algebraic factorizations (c) and (d).

(A2.3) Now we indicate how one can do the arithmetic factorization for deformation geometry, i.e., the Weil-Petersson metric. To this end, we need to get a simliar result as in Theorem A1.3.1. More precisely, the following statement should be

first established.

With the same notation as above, the resulting metric on the relative logarithmic canonical line bundle for $\overline{\pi_{q,N}}$ obtaining from the standard (singular) hyperbolic metrics on the fibers is continuous and good.

Indeed, as pointed out by Professor Fujiki, this result may be obtained as a direct consequence of Theorem A1.3.1 of Wolpert by using the structure of the universal curve over $\overline{\mathcal{M}_{q,N}}$.

With this, in particular, we will obtain a continuous metric on the Deligne pairing $\langle K(\mathbb{P}_1 + \cdots + \mathbb{P}_N), K(\mathbb{P}_1 + \cdots + \mathbb{P}_N) \rangle$. That is, we have a continuous metrized line bundle $\langle \underline{K(\mathbb{P}_1 + \cdots + \mathbb{P}_N)}_{\text{hyp}}, \underline{K(\mathbb{P}_1 + \cdots + \mathbb{P}_N)}_{\text{hyp}} \rangle$ on $\overline{\mathcal{M}_{q,N}}$. From here, by using the property of Deligne metric on Deligne pairings, we can further conclude that in Masur's result, e.g., Theorem A1.1.1(iv), the positive metrix is really nothing but the one coming from the Weil-Petersson metric for $\mathcal{M}_{q-1,N+2}$. So the Weil-Petersson metric factors extremely well.

(A2.4) To give the arithmetic factorization for Takhtajan-Zograf metric, we need first decompose them into N -pieces. In fact, by symmetry, we see that

$$c_1 \left(\langle \underline{K}_{\text{hyp}}, \underline{\mathbb{P}_{i_{\text{hyp}}}} \rangle \otimes \langle \underline{K(\mathbb{P}_1 + \cdots + \mathbb{P}_N)}_{\text{hyp}}, \underline{\mathbb{P}_{i_{\text{hyp}}}} \rangle \right) = \frac{4}{3} \omega_{\text{WZ}}^{(i)}, \quad i = 1, \dots, N.$$

Here, for $i = 1, \dots, N$, $\omega_{\text{WZ}}^{(i)}$ denotes the i -th Takhtajan-Zograf Kähler form associated to the i -th Takhtajan-Zograf metric on $\mathcal{M}_{q,N}$ defined by using the i -th Eisenstein series. Thus we should consider line bundles $\langle K, \mathbb{P}_i \rangle \otimes \langle K(\mathbb{P}_1 + \cdots + \mathbb{P}_N), \mathbb{P}_i \rangle$, $i = 1, \dots, N$. By algebraic factorization (b) and (c) in (A2.1), we see that they factor into $\left(\langle \tilde{K}, \tilde{\mathbb{P}}_i \rangle \otimes \langle \tilde{K}(\tilde{\mathbb{P}}_1 + \cdots + \tilde{\mathbb{P}}_N + \mathbb{R} + \mathbb{S}), \tilde{\mathbb{P}}_i \rangle \right) \otimes \left(\langle \mathbb{R} + \mathbb{S}, \tilde{\mathbb{P}}_i \rangle \right)$, $i = 1, \dots, N$. At this moment, I should say that arithmetically, the appearance of $\langle \mathbb{R} + \mathbb{S}, \tilde{\mathbb{P}}_i \rangle$ is extremely unpleasant, as I cannot show that arithmetically it is trivial. (Indeed, I think it is hardly the case.) But the line bundle $\langle \mathbb{R} + \mathbb{S}, \tilde{\mathbb{P}}_i \rangle$ is trivial, so let us for the time being pretend that such an appearance is harmless for the discussion follows.

As for the case about the Weil-Petersson metric, to understand the arithmetic factorization, we need to study the corresponding metrics on line bundles over $\overline{\mathcal{C}_{q,N}}$ first. In (6.1), we already introduce natural metrics on K_M and on P_i , $i = 1, \dots, N$ for each punctured Riemann surface by introducing an invariant called Arakelov-Poincaré volume. The point now is whether such metrics will form continuous metrics on K and \mathbb{P}_i , $i = 1, \dots, N$, when we are working on a family. By looking at

the result of To and myself recalled in (A1.4), it is resonable to conclude that if the total volume of M with respect to the metric induced from $\underline{K}_{M_{\text{hyp}}}$ is an absolute constant, i.e., the associated total volume is independent of M and P_1, \dots, P_N , we should then can glue these metrics together to get continuous metrics on K and \mathbb{P}_i , $i = 1, \dots, N$ on $\overline{\mathcal{C}_{q,N}}$. For this latest purpose, we next give the final main result of this paper.

Theorem A2.4.1. *With the same notation as above, the total volume of M for the metric induced from $\underline{K}_{M_{\text{hyp}}}$ is a constant depending only on q and N .*

Proof. This is a direct consequence of the geometric interpretation of our determinant metric on $\underline{\lambda}_{1_{\text{hyp}}}$. Indeed, if we denote the total volume of M for the metric induced from $\underline{K}_{M_{\text{hyp}}}$ by $A(M; \underline{K}_{M_{\text{hyp}}})$, then up to a constant depending only on q and N , the inner product of our determinant metric for the generator $1 \otimes (\phi_1 \wedge \dots \wedge \phi_q)^\vee$ of λ_1 is nothing but $A(M; \underline{K}_{M_{\text{hyp}}})$ times the inverse $\det^* \Delta_{\text{hyp}}$. Here, as before, we denote $\{\phi_i\}_{i=1}^q$ an orthonormal basis of $H^0(K_M)$ with respect to the natural pairing. But we know that $\det^* \Delta_{\text{hyp}}$ up to a constant depending only on q and N is simply $Z'(1)$ with $Z(s)$ denoting the corresponding Selberg zeta function, by the fact that $Z(s)$ has a simple zero at $s = 1$. Thus by the curvature formula for $\underline{\lambda}_{1_{\text{hyp}}}$, we see that

$$dd^c A(M; \underline{K}_{M_{\text{hyp}}}) = 0,$$

when we move M in $\mathcal{M}_{g,N}$. But in $\overline{\mathcal{M}_{g,N}}$, locally, the absolute horizontal boundary and the relative horizontal boundaries, i.e., the fake diagonal divisors Δ_S can be contracted. (I learn this from Prof. Fujiki on the way to Kinoshita.) This completes the proof.

Remark A2.4.1. Indeed, we would like to guess that $A(M; \underline{K}_{M_{\text{hyp}}}) = 2\pi(2q - 2)$. But for the time being, it seems to be quite impossible to prove this, as we need more precise degeneration information for the quantities introduced in this paper. On the other hand, if this is true, then there is a great chance to simplify the discussion in §6 and §7.

Remark A2.4.2. The reader should know that the metric defined on $\underline{K}_{M_{\text{hyp}}}$ is obtained in an arithmetic manner: we first use the hyperbolic Green's function and the associated beta function to define an hyperbolic Arakelov metric on K_M , which is quite suitable for our arithmetic purpose; then we multiple this metric by a highly transcendental invariant, the so-called Arakelov-Poincaré volume to obtain the metric, which is motivated by the work of D'Hoker-Phong and Sarnak. So the

possible geometric definition of this metric, proposed in the previous remark, on K_M would be very interesting.

With the above, we may assume that there are globally defined metrized line bundles $\underline{K}_{\text{hyp}}$ and $\underline{\mathbb{P}}_{i_{\text{hyp}}}$, $i = 1, \dots, N$, with continuous metrics on $\overline{\mathcal{C}_{q,N}}$. Hence we further get metrized line bundles $\langle \underline{K}_{\text{hyp}}, \underline{\mathbb{P}}_{i_{\text{hyp}}} \rangle \otimes \langle \underline{K(\mathbb{P}_1 + \dots + \mathbb{P}_N)}_{\text{hyp}}, \underline{\mathbb{P}}_{i_{\text{hyp}}} \rangle$, $i = 1, \dots, N$ with continuous metrics on $\overline{\mathcal{M}_{q,N}}$. This clearly shows that the Takhtajan-Zograf metrics $\langle \cdot, \cdot \rangle_{\text{TZ}}^{(i)}$, $i = 1, \dots, N$, have natural factorizations, which is exactly the same as what has happened for Weil-Petersson metrics. In particular, we see that the degeneration of the i -th Eisenstein series will correspond to exactly the i -th Einstein series of the central fiber. We would like to point out that such a degeneration has been studied by others, notably, Wolpert ([Wo3]). But the picture drew here by using the arithmetic factorization seems to be more clear than what is obtained before. In fact, we see that the additional Eisenstein series corresponding to new punctures R and S can never be the limit of the original Eisenstein series from nearby fibers. Nevertheless, by the arithmetic factorization for the determinant line bundles $\underline{\lambda}_{n_{\text{hyp}}}$, which may now be obtained by using the above arithmetic factorization of arithmetic intersection via the generalized Mumford type isometries, we see that the additional Eisenstein series corresponding to new punctures R and S are obtained from the spectrum degeneration via Selberg zeta functions. So instead of traditionally studying the degeneration of the combination of the Selberg zeta function and the small eigen-values, one may directly study the degeneration of the Selberg zeta function itself, as we expect this will give additional information for additional Eisenstein series corresponding to new punctures R and S .

(A2.5) We conclude this appendix and hence this paper by the following remark. In the paper, we offer a way to understand the global geometry of a general Riemann surface. Undoubtedly, this is just the beginning of the story. Personally, I believe that with the arithmetic factorization proposed here one may finally find an alternative way to understand the arithmetic Miyaoka-Yau inequality, if we take Belyi's theorem [Be] and Hilbert's irreducibility theorem [La1] into consideration, instead of trying to establish a p -adic deformation theory, if it exists.

REFERENCES

- [Ai] W. Aitken: *An arithmetic Riemann-Roch theorem for singular arithmetic surfaces*, Memoirs of the AMS, No. **573**, 1996
- [Ar] S. Arakelov: Intersection theory of divisors on an arithmetic surface, *Izv. Akad. Nauk SSSR Ser. Mat.*, 38 No. **6**, (1974)
- [BM] A. Beilinson and V. Ginsburg, Infinitesimal structure of moduli spaces of G -bundles, *IMRN, Duke J. Math.* **4** (1992), 63-74
- [BM] A. Beilinson and Y. Manin, The Mumford form and the Polyakov measure in string theory, *Comm. Math. Phys*, **107**, 359-376, (1986)
- [Be] G. Belyi, On Galois extensions of a maximal cyclotomic field, *Math. USSR Izv.* **14** (1980), 247-256
- [De1] P. Deligne: *Equations différentielles a points singuliers réguliers*, Lecture Notes in Math. **163**, Berlin-Heideberg-New York, Springer, (1970)
- [De2] P. Deligne: Le déterminant de la cohomologie, *Current trends in arithmetic algebraic geometry*, Contemporary Math. Vol. **67**, 93-178, (1987)
- [DM] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, *IHES Publ. Math.* **36** (1969), 75-109
- [D'HP] E. D'Hoker and D.H. Phong, On determinant of Laplacians on Riemann surfaces, *Comm. Math. Physics*, **104**, 537-545, (1986)
- [Ef] I. Efrat, determinant of Laplacians on surfaces of finite volume, *Commun. Math. Phys*, **119**, 443-451, (1988)
- [El] R. Elkik, Métriques sur les fibrés d'intersection, *Duke Math. J.* **61**, 303-328, (1990)
- [Fa] G. Faltings: Calculus on Arithmetic Surfaces, *Ann. Math.* **119**, 387-424, (1984)
- [Fay1] J. Fay, *Theta functions on Riemann surfaces*, Lecture notes in Math. **352**. Berlin-Heideberg-New York, Springer, (1973)
- [Fay2] J. Fay, *Kernel functions, analytic torsion, and moduli spaces*, Memoirs of AMS, No. **464**, 1992
- [Fu] A. Fujiki, An L^2 Dolbeault lemma and its applications, *Publ. RIMS*, **28**, 845-884

(1992)

- [FM] W. Fulton and R. MacPherson, A compactification of configuration spaces, *Ann. Math.* (2), **139** (1994), no. 1, 183-225
- [Jo] J. Jorgenson, Asymptotic behavior of Faltings delta function. *Duke Math. J.* **61** (1990), 221-254
- [JL1] J. Jorgenson and R. Lundelius, Convergence theorems for relative spectral functions on hyperbolic Riemann surfaces with hyperbolic cusps, *Duke Math. J.* **80**, 785-819, (1995)
- [JL2] J. Jorgenson and R. Lundelius, Continuity of relative hyperbolic spectral theory through metric degeneration, *Duke Math. J.* **84**, 47-81, (1997)
- [Kn] F. Knudsen, The projectivity of the moduli space of stable curves, II, and III, *Math. Scand.* **52**(1983), 161-199, 200-212
- [KM] F. Knudsen and D. Mumford, The projectivity of the moduli space of stable curves, I. *Math. Scand.* **39** (1976), 19-55
- [La1] S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1983
- [La2] S. Lang, *Introduction to Arakelov theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1988
- [Ma] H. Masur, The extension of the Weil-Petersson metric to the boundary of Teichmüller space, *Duke Math. J.*, **43** (1976), 623-635
- [MB] L. Moret-Bailly: Le formulae de Noether pour les surfaces arithmétiques, *Invent. Math.*, (1990)
- [Mu1] D. Mumford, Hirzebruch's proportionality theorem in the non-compact case, *Invent. Math.* (1977)
- [Mu2] D. Mumford, Stability of projective varieties, *L'Ens. Math.*, **24** (1977), 39-110
- [Mu3] D. Mumford, Towards an Enumerative geometry of the moduli space of curves, in *Arithmetic geometry*, dedicated to Shafarevich on his 60th birthday, 271-328, (1983)
- [Qu] D. Quillen, determinant of Cauchy-Riemann operators over Riemann surfaces,

- Func. Anal. Appl. **19**, 31-34 (1986)
- [RS] D. Ray and I. Singer, Analytic torsion for complex manifolds, Ann. Math. **98**, 154-177 (1973)
- [Sa] P. Sanark, determinant of Laplacians, Commun. Math. Phys. **110**, 113-120 (1987)
- [So] Ch. Soulé, Géométrie d'Arakelov des surfaces arithmétiques. Séminaire Bourbaki, Vol. 1988/89. Astérisque No. **177-178** (1989), Exp. No. 713, 327-343
- [TW1] W.-K. To and L. Weng, The asymptotic behavior of Green's functions for quasi-hyperbolic metrics on degenerating Riemann surfaces, Manuscripta Math. **93**, 465-480 (1997)
- [TW2] W.-K. To and L. Weng, Admissible Hermitian metrics on families of line bundles over degenerating Riemann surfaces, preprint, 1998
- [TZ1] L. Takhtajan and P. Zograf, The Selberg zeta function and a new Kähler metric on the moduli space of punctured Riemann surfaces, J. Geom. Phys. **5**, 551-570 (1988)
- [TZ2] L. Takhtajan and P. Zograf, A local index theorem for families of $\bar{\partial}$ -operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces, Commun. Math. Phys. **137**, 399-426 (1991)
- [We1] L. Weng, Ω -admissible theory, to appear in the Proceedings of London Math. Soc.
- [We2] L. Weng, *Hyperbolic Metrics, Selberg Zeta Functions and Arakelov Theory for Punctured Riemann Surfaces*, Lecture Notes at Osaka University, manuscript, Osaka University (1998)
- [Wo1] S. Wolpert, Chern forms and the Riemann tensor for the moduli space of curves, Inv. Math. **85**, 119-145 (1986)
- [Wo2] S. Wolpert, The hyperbolic metric and the geometry of universal curve. J. Differ. Geom. **31** (1990), 417-472
- [Wo3] S. Wolpert, Spectral limits for hyperbolic surfaces, I, II, Invent. Math. **108** (1992), 67-90, 91-129